Mech Major – Past Paper Pack 1

NOTE: This should be used in addition to the Mech Minor papers.

Questions made from MEI papers between 2003 and 2005

This does not contain any questions on projectiles or motion under a variable force.

1.

Two identical light springs have natural length 1 m and stiffness 400 N m^{-1}. One is suspended vertically with its upper end fixed to a ceiling and a particle of mass 2 kg hanging in equilibrium from its lower end.

(i) Calculate the extension of the spring. [2]

The second spring is then attached to the particle and its other end is attached to the floor. The system is in equilibrium with the two springs in the same vertical line. The distance between the floor and ceiling is 2.5 m and the extension of the upper spring is e metres. This situation is shown in Fig. 1.

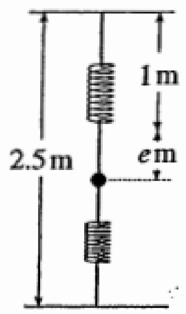


Fig. 1

(ii) Write down the extension of the lower spring in terms of e. [1]

(iii) Write down the equilibrium equation for the particle and hence calculate e. [4]

(iv) Calculate the tensions in the springs. [2]

The particle is now pulled vertically downwards a short distance and released.

(v) Show that the particle performs simple harmonic motion with period $\frac{1}{10}\pi$ seconds. [6]

2.

Part of the track of a theme park ride consists of a vertical drop, AB, of 62.5 m and the arc, BC, of a circle with centre O and radius 25 m. BO is horizontal. Fig. 3.1 shows the situation with the car in its initial position. Fig. 3.2 shows the car in a later position when it has turned through an angle θ about O.

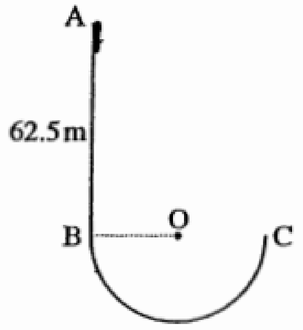


Fig. 3.1

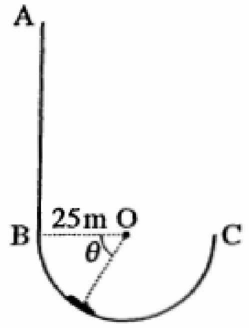


Fig. 3.2

The car is modelled as a particle of mass m kg dropped from rest at A. All resistance forces are neglected.

(i) Show that the speed of the car at B is $35\,\text{m}\,\text{s}^{-1}$. [1]

When the car has turned through an angle θ, its speed is $v\,\text{m}\,\text{s}^{-1}$.

(ii) Find an expression for v^2 in terms of θ and g and hence show that the force, R N, of the track on the car is given by $R = mg(3\sin\theta + 5)$. [8]

(iii) Find expressions for the radial and tangential components of the acceleration in terms of θ and g. Hence show that the magnitude of the acceleration, in $\text{m}\,\text{s}^{-2}$, is

$$g\sqrt{26 + 20\sin\theta + 3\sin^2\theta}.$$ [6]

3.

You are given that:

Centres of mass

Triangular lamina: $\frac{2}{3}$ along median from vertex

Solid hemisphere, radius r: $\frac{3}{8}r$ from centre

A uniform solid is made by rotating the region in the first quadrant between the curve $y = \sqrt{1-x}$ and the coordinate axes through one revolution about the y-axis, as shown in Fig. 4.1.

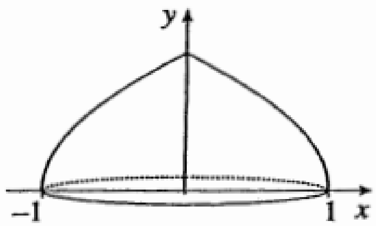


Fig. 4.1

(i) Show that the centre of mass of the solid is at $(0, \frac{5}{16})$. [7]

Fig. 4.2 shows a buoy made by attaching a uniform solid of the above shape and of mass m to a uniform solid hemisphere of radius 1 and mass λm. The buoy has its axis of symmetry vertical.

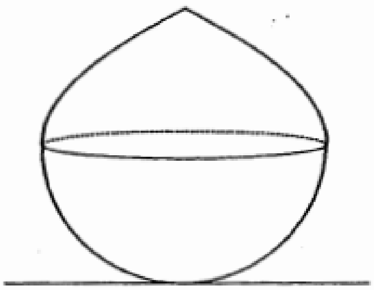


Fig. 4.2

(ii) Show that the distance of the centre of mass of the buoy above its lowest point is

$$\frac{10\lambda + 21}{16(\lambda + 1)}.$$ [4]

[You may assume the standard result for the centre of mass of a hemisphere.]

(iii) The buoy is placed with a point of the hemisphere on horizontal ground. The axis of symmetry is not vertical. Find the range of values of λ for which the buoy will tend to right itself. [4]

4.

Spheres A, B and C of masses 2 kg, 3 kg and 5 kg, respectively, are at rest on a smooth, horizontal table, as shown in Fig. 1. The spheres move only in the same straight line and all impacts on the table are direct.

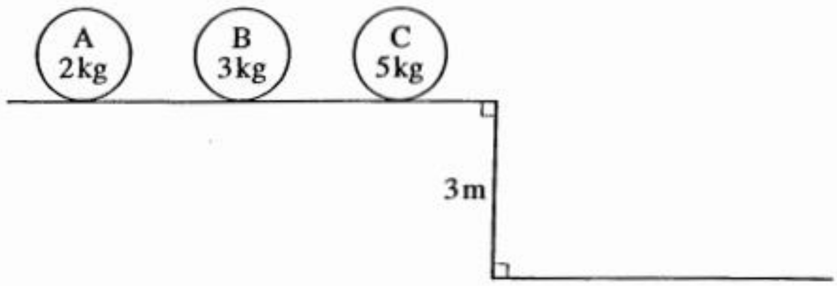


Fig. 1

An impulse of 10 N s is applied to sphere A in the direction A to B.

(i) Find the speed with which the sphere moves off. [1]

Sphere A now collides and coalesces with sphere B to form the object AB.

(ii) Show that the speed of the object AB after the collision is $2\,\mathrm{m\,s^{-1}}$. [2]

Object AB now collides with sphere C. The coefficient of restitution in this collision is 0.6.

(iii) Show that the speed of sphere C after the collision is $1.6\,\mathrm{m\,s^{-1}}$. [5]

Sphere C leaves the table and should now be regarded as a projectile subject to negligible air resistance. Sphere C bounces off the smooth horizontal floor that is 3 m below the table top, as shown in Fig. 1. The coefficient of restitution in this collision is also 0.6.

(iv) Show that sphere C bounces off the floor with a vertical component of velocity of $\sqrt{2.16g}\,\mathrm{m\,s^{-1}}$.

Calculate the angle to the horizontal with which sphere C leaves the floor. [6]

[Total 14]

5.

A brass pendulum consists of a rod AB freely hinged at the end A with a sphere at the end B, as shown in Fig. 1.

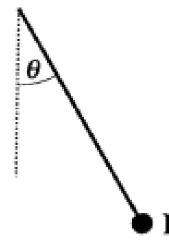


Fig. 1

When oscillating, the total energy, E, of the pendulum is given by the equation

$$E = \tfrac{1}{2}I\omega^2 - mgh\cos\theta,$$

where ω is the angular speed, m is the mass of the pendulum, h is the distance of the centre of mass of the pendulum from A, θ is the angle the pendulum makes with the downward vertical and I is a quantity known as the moment of inertia of the pendulum.

(i) Use this equation to deduce the dimensions of I. [4]

It is suggested that the period, T, of the pendulum is given by $T = kI^{\alpha}(mg)^{\beta}h^{\gamma}$ where k is a dimensionless constant.

(ii) Use dimensional analysis to find α, β and γ. [5]

For a particular pendulum, the equation of motion can be shown to be

$$\ddot{\theta} + 9\sin\theta = 0.$$

(iii) Show that, for small θ, the motion is approximately simple harmonic. [2]

The time, t seconds, is recorded from an instant when the pendulum is observed to be at an angle of 0.025 radians to the vertical and moving away from the vertical. The amplitude of the oscillations is observed to be 0.05 radians.

(iv) Find θ in terms of t and hence find the value of t when the pendulum first passes through the vertical. [4]

6.

A railway engine of mass 50 tonnes travels at $30\,\text{m}\,\text{s}^{-1}$ along a section of horizontal track around a circular bend of radius 500 m.

(i) Calculate the lateral (sideways) force exerted on the rails by the engine, indicating clearly where you use Newton's second and third laws. [4]

Subsequently, for safety reasons, it is decided that the lateral force on the rails should not exceed 50 000 N. One way to ensure this is to impose a speed limit of $V\,\text{m}\,\text{s}^{-1}$, so that at this speed the force is 50 000 N.

(ii) Calculate V. [2]

Another way to restrict the force on the rails and allow the engine to travel at $30\,\text{m}\,\text{s}^{-1}$ is to bank the track at an angle θ to the horizontal. This is shown in Fig. 2, where AB is a line of greatest slope.

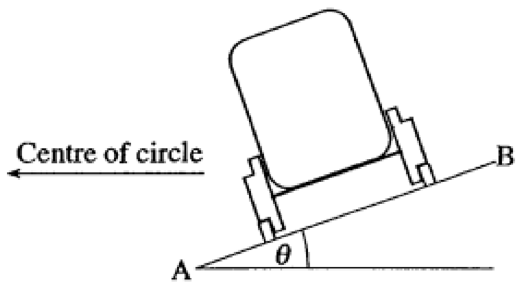


Fig. 2

(iii) When the engine travels at this speed, find, in terms of θ, the component of acceleration in the direction BA. Hence, or otherwise, show that the lateral force, T N, is given by

$$T = 90\,000\cos\theta - 490\,000\sin\theta. \qquad [5]$$

(iv) Express T in the form $R\cos(\theta + \alpha)$. Hence calculate θ when $T = 50\,000$. [4]

7.

A light elastic string AB, of natural length 0.8 m and modulus 5 N, is attached at A to a ceiling which is 2.4 m above the floor. A small ball of mass 0.2 kg is attached to the other end B and hangs in equilibrium.

(i) Calculate the length of the string. [3]

(ii) The ball is pulled down until it touches the floor with AB vertical and it is then released from rest. Calculate the speed at which it hits the ceiling. [4]

A second light elastic string of modulus 5 N and natural length l_1 m, where $l_1 < 2.4$, is attached to the ball at B and to the floor vertically below A. The ball is held at rest on the floor with AB vertical and it is then released.

(iii) Find the range of values of l_1 for which the ball will still hit the ceiling. [8]

8.

A uniform lamina is in the shape of a triangle OAB, where O is the origin and A and B have coordinates (b, h) and $(b, 0)$ respectively, as shown in Fig. 4.

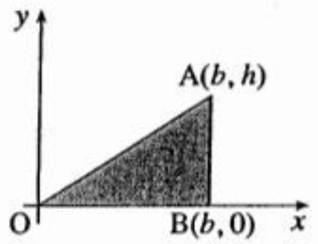


Fig. 4

(i) Find the equation of the line through O and A. Hence show by integration that the y-coordinate of the centre of mass of the lamina is $\frac{1}{3}h$.

Write down the x-coordinate of the centre of mass. [6]

A second triangular lamina, OBC, has vertices at O, B and the point C with coordinates (c, h) where $0 < c < b$.

(ii) Show that the centre of mass of OBC is at the point $(\frac{1}{3}(b + c), \frac{1}{3}h)$. [5]

A uniform triangular prism with cross-section OBC is placed on a rough plane inclined at 30° to the horizontal. The cross-section OBC is vertical with OB along a line of greatest slope and B higher than O.

(iii) Find, in terms of b and h, the range of values of c for which the prism will topple. Deduce that the prism will not topple if $h < b\sqrt{3}$. [4]

9.

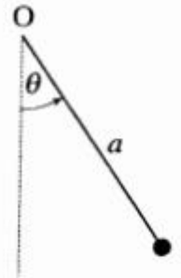


Fig. 2.1

Fig. 2.1 shows a particle of mass m suspended from a fixed point O by means of a light, inextensible string of length a. The particle moves in a vertical circle centre O. The speed of the particle is u at the lowest point of its motion. When the string makes an angle θ with the downward vertical the speed of the particle is v. Resistances to motion may be neglected.

(i) By considering the energy of the particle, show that

$$v^2 = u^2 - 2ag(1 - \cos\theta).$$ [3]

(ii) Find the tension in the string in terms of a, g, m, u and θ. [5]

(iii) Show that the condition for the particle to perform complete circles is $u^2 \geqslant 5ag$. [3]

The particle is now released from rest at the same level as O with the string taut. As the particle swings, the string hits a small peg a distance b vertically below O, as shown in Fig. 2.2, and wraps round it.

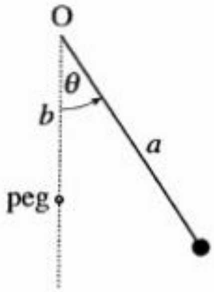


Fig. 2.2

(iv) The particle just completes a vertical circle after the string wraps round the peg. Find an expression for b in terms of a. [4]

[Total 15]

10.

A sketch of the curve with equation $x = \sqrt{2y^2 - y^4 + 1}$ for $-1 \leqslant y \leqslant 1$ is shown in Fig. 3.1.

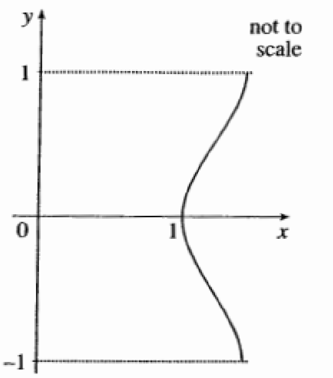


Fig. 3.1

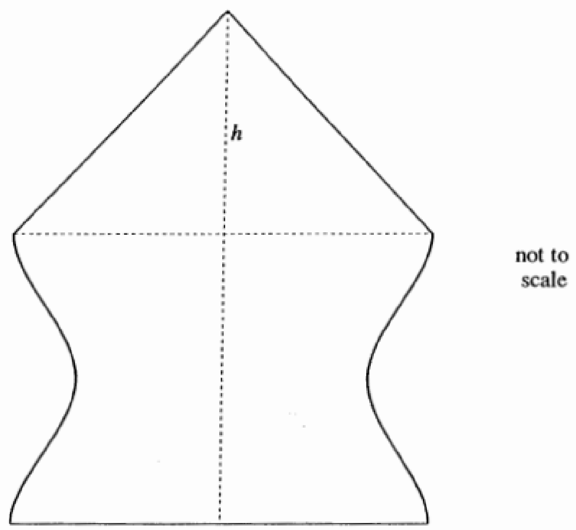
A uniform solid, S, is in the shape formed by rotating the region bounded by the curve $x = \sqrt{2y^2 - y^4 + 1}$, the lines $y = \pm 1$ and the y-axis through 2π radians about the y-axis. The units of the axes are metres.

(i) Show that the volume of S is $\frac{44}{15}\pi$ m^3.

Show that the curve $x = \sqrt{2y^2 - y^4 + 1}$ is symmetrical about the line $y = 0$ and hence write down the coordinates of the centre of mass of S. [5]

(ii) Using integration, show that the centre of mass of a uniform right circular solid cone of vertical height h and base radius r is at a distance of $\frac{1}{4}h$ from the plane face along the axis of symmetry of the cone. [You may assume that the volume of this cone is $\frac{1}{3}\pi r^2 h$.] [5]

A right circular cone has height h and its plane face has the same radius as a plane face of S. A uniform solid, T, is formed by attaching this cone to S so that the plane faces meet and the axes of symmetry of S and the cone are in the same line, as shown in Fig. 3.2.



cross-section of T

Fig. 3.2

(iii) Calculate the value of h if the centre of mass of T lies in the plane of the join of S with the cone. [5]

[Total 15]

11.

A light spring of stiffness 19.6 N m^{-1} is fixed at its bottom end and slides freely in a vertical tube. Initially, the spring is compressed 0.05 m by a disc of mass m kg resting on it in equilibrium, as shown in Fig. 4.1.

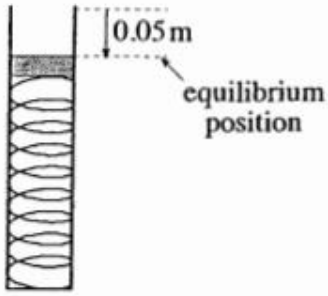


Fig. 4.1

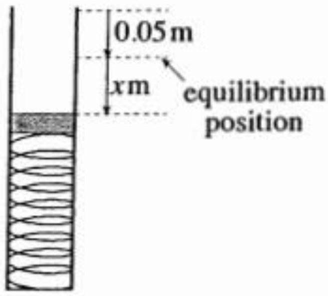


Fig. 4.2

(i) Calculate the value of m. [2]

The disc is pushed down a distance 0.1 m from the equilibrium position and released from rest.

(ii) Show that the disc begins to move in simple harmonic motion with equation $\ddot{x} + 196x = 0$, where x m is the displacement of the disc below its equilibrium position, as shown in Fig. 4.2. [5]

The disc is in contact with the spring until the spring reaches its natural length. At this instant, the spring is stopped from moving and the disc loses contact with it.

(iii) Calculate the speed of the disc as it leaves the spring.

Calculate also the time after its release for which the disc is in contact with the spring. [6]

With the disc resting on the spring in equilibrium, the disc is given a velocity of v m s^{-1} downwards. The system comes instantaneously to rest 0.2 m below its original rest position.

(iv) Calculate v. [3]

[Total 16]

12.

Each of two light elastic strings, AB and BC, has modulus 20 N. AB has natural length 0.5 m and BC has natural length 0.8 m. The strings are both attached at B to a particle of mass 0.75 kg. The ends A and C are fixed to points on a smooth horizontal table such that AC = 2 m, as shown in Fig. 1.

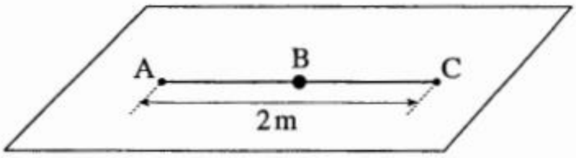


Fig. 1

Initially the particle is held at the mid-point of AC and released from rest.

(i) Find the tension in each string before release and calculate the acceleration of the particle immediately after it is released. [5]

The particle is now moved to the position where it is in equilibrium. The extension in AB is e m.

(ii) Calculate e. [4]

The particle is now held at A and released from rest.

(iii) Show that in the subsequent motion BC becomes slack. Calculate the furthest distance of the particle from A. [6]

13.

A simple pendulum consists of a light inextensible string AB of length l with the end A fixed and a particle of mass m attached to B. The pendulum oscillates with period T.

(i) It is suggested that T is proportional to a product of powers of m, l and g. Use dimensional analysis to find this relationship. [4]

The angle that the string makes with the downward vertical at time t is θ. The particle is released from rest with the string taut and $\theta = \theta_0$.

(ii) Use the equation of motion of the particle to find the angular acceleration, $\ddot{\theta}$, in terms of θ, l and g. [3]

The angle θ_0 is chosen so that θ remains small throughout the motion.

(iii) Use the small angle approximation for $\sin\theta$ to show that the particle performs approximate angular simple harmonic motion. State the period of the motion and verify that it is consistent with your answer to part **(i)**. [4]

(iv) Calculate the proportion of time for which the particle travels faster than half of its maximum speed. [4]

14.

Michael is attempting to make a small car do a 'loop-the-loop' on a smooth toy racing track. He propels a car of mass m kg towards a section of the track in the form of a vertical circle of radius 0.2 m and the car enters the circle at its lowest point with a speed of $2.8\,\text{m}\,\text{s}^{-1}$. During the motion around the circle the angle the car has turned through is denoted by θ, as shown in Fig. 3.

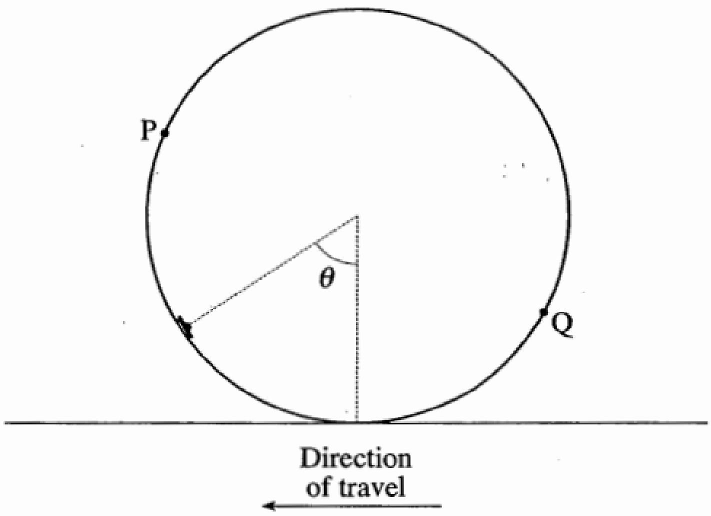


Fig. 3

(i) Show that the speed, $v\,\text{m}\,\text{s}^{-1}$, of the car is given by $v^2 = 3.92(1 + \cos\theta)$. Hence show that the reaction of the track on the car, R N, is given by $R = 9.8m(2 + 3\cos\theta)$. [7]

The car leaves the track at the point P where $\theta = \alpha$.

(ii) Calculate α. [2]

(iii) Calculate the speed of the car at P and hence calculate the greatest height of the car above the level of P. [3]

The car hits the track at the point Q which is $\frac{22}{135}$ m below the level of the centre of the circle.

(iv) Calculate the speed with which the car hits the track at Q. [3]

15.

Fig. 4.1 shows a uniform lamina OAB in the shape of the region between the curve $y = 4x - x^3$ and the x-axis. The point G is the centre of mass of the lamina.

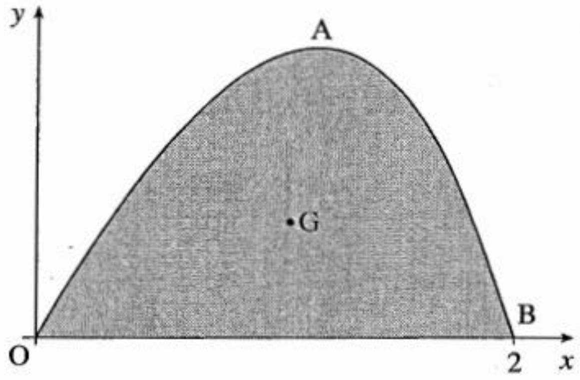


Fig. 4.1

(i) Show that G has coordinates $\left(\frac{16}{15}, \frac{128}{105}\right)$. [11]

OAB is suspended by wires at O and B and hangs in equilibrium in a vertical plane with OB horizontal. The wire at B is at 60° to the horizontal and the wire at O is at $\alpha°$ to the horizontal, as shown in Fig. 4.2.

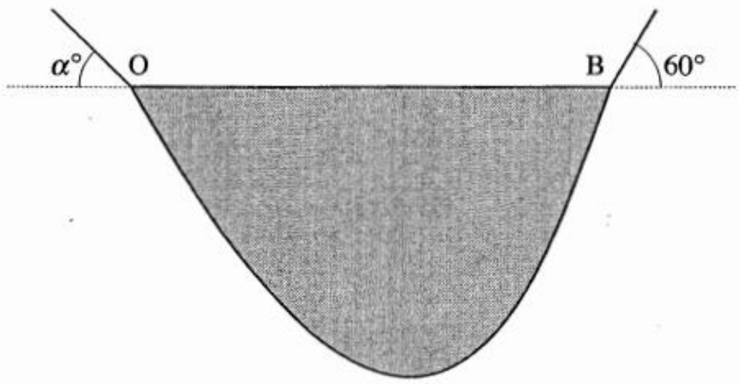


Fig. 4.2

(ii) Calculate α. [4]

16.

(a) The speed, v, of a particle whose motion is simple harmonic is given by

$$v^2 = \omega^2(a^2 - x^2),$$

where x is the displacement from the centre of the motion, a is the amplitude and ω is a further constant.

(i) State the dimensions of a, x and v and hence establish the dimensions of ω. [3]

(ii) Name a quantity with the same dimensions as ω. [1]

(b) A fisherman's float of length 0.15 m has a mass of 0.015 kg and moves in a vertical line. It is at rest in equilibrium when the bottom of the float is 0.098 m under water.

In a general position, the bottom of the float is y m under water and is x m below the equilibrium position, as shown in Fig. 2. The upward force on the float, F N, is directly proportional to y. You may assume that the float is never completely immersed.

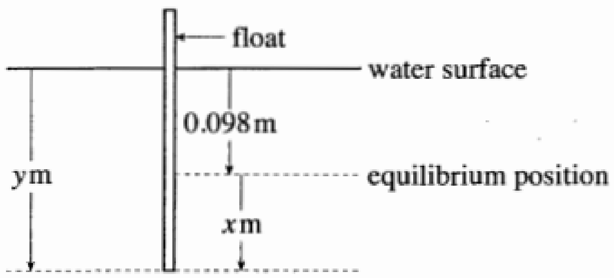


Fig. 2

(i) Show that $F = 1.5y$. [2]

The float is pulled down below the equilibrium position and released from rest.

(ii) By using Newton's second law, show that the equation of motion is $\ddot{x} + 100x = 0$. [4]

The float is released from rest 0.02 m below its equilibrium position.

(iii) Calculate the speed of the float when it is 0.01 m above its equilibrium position and moving **downwards**. Calculate also the time taken to reach this position. [5]

[Total 15]

17.

A smooth, horizontal circular disc of radius 0.4 m has a rough vertical rim. Initially, the disc is rotating about its centre with a constant angular speed of $10\,\text{rad}\,\text{s}^{-1}$ with a particle of mass 0.3 kg touching the rim, as shown in Fig. 3. The coefficient of friction between the particle and the rim is μ and the particle does not slip.

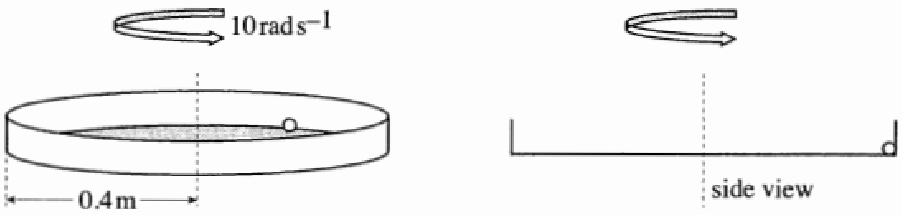


Fig. 3

(i) Explain briefly why there is no horizontal frictional force between the particle and the rim. [1]

(ii) Calculate the normal reaction of the rim on the particle. [2]

(iii) Find the least possible value of μ so that the particle could travel in a horizontal circle in contact with the rim but not in contact with the horizontal disc. [3]

The disc is now accelerated. All points on the rim have a tangential acceleration of $2\,\text{m}\,\text{s}^{-2}$ so that the angular speed, t seconds after the acceleration starts, is $(10 + 5t)\,\text{rad}\,\text{s}^{-1}$. The particle is in contact with the smooth horizontal disc and with the rim. The vertical frictional force is zero.

(iv) Show that the normal reaction, R N, of the rim on the particle at time t is given by

$$R = 3(2 + t)^2.$$ [2]

(v) Show that, in order to prevent slipping at time t,

$$\mu \geqslant \frac{1}{5(2 + t)^2}.$$

Deduce that the particle will not slip if $\mu \geqslant 0.05$, explaining your reasoning clearly but briefly. [5]

[Total 13]