Groups Past Paper Question Pack

MEI FP3 2006

The group G consists of the 8 complex matrices $\{\mathbf{I}, \mathbf{J}, \mathbf{K}, \mathbf{L}, -\mathbf{I}, -\mathbf{J}, -\mathbf{K}, -\mathbf{L}\}$ under matrix multiplication, where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{J} = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix}.$$

(i) Copy and complete the following composition table for G. [6]

	I	**J**	**K**	**L**	**–I**	**–J**	**–K**	**–L**
I	**I**	**J**	**K**	**L**	**–I**	**–J**	**–K**	**–L**
J	**J**	**–I**	**L**	**–K**	**–J**	**I**	**–L**	**K**
K	**K**	**–L**	**–I**					
L	**L**	**K**						
–I	**–I**	**–J**						
–J	**–J**	**I**						
–K	**–K**	**L**						
–L	**–L**	**–K**						

(Note that $\mathbf{JK} = \mathbf{L}$ and $\mathbf{KJ} = -\mathbf{L}$.)

(ii) State the inverse of each element of G. [3]

(iii) Find the order of each element of G. [3]

(iv) Explain why, if G has a subgroup of order 4, that subgroup must be cyclic. [4]

(v) Find all the proper subgroups of G. [5]

(vi) Show that G is not isomorphic to the group of symmetries of a square. [3]

MEI FP3 2007

(i) Prove that, for a group of order 10, every proper subgroup must be cyclic. [4]

The set $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ is a group under the binary operation of multiplication modulo 11.

(ii) Show that M is cyclic. [4]

(iii) List all the proper subgroups of M. [3]

The group P of symmetries of a regular pentagon consists of 10 transformations

$$\{A, B, C, D, E, F, G, H, I, J\}$$

and the binary operation is composition of transformations. The composition table for P is given below.

	A	B	C	D	E	F	G	H	I	J
A	C	J	G	H	A	B	I	F	E	D
B	F	E	H	G	B	A	D	C	J	I
C	G	D	I	F	C	J	E	B	A	H
D	J	C	B	E	D	G	F	I	H	A
E	A	B	C	D	E	F	G	H	I	J
F	H	I	D	C	F	E	J	A	B	G
G	I	H	E	B	G	D	A	J	C	F
H	D	G	J	A	H	I	B	E	F	C
I	E	F	A	J	I	H	C	D	G	B
J	B	A	F	I	J	C	H	G	D	E

One of these transformations is the identity transformation, some are rotations and the rest are reflections.

(iv) Identify which transformation is the identity, which are rotations and which are reflections. [4]

(v) State, giving a reason, whether P is isomorphic to M. [2]

(vi) Find the order of each element of P. [3]

(vii) List all the proper subgroups of P. [4]

MEI FP3 2008

A binary operation $*$ is defined on real numbers x and y by

$$x * y = 2xy + x + y.$$

You may assume that the operation $*$ is commutative and associative.

(i) Explain briefly the meanings of the terms 'commutative' and 'associative'. [3]

(ii) Show that $x * y = 2\left(x + \frac{1}{2}\right)\left(y + \frac{1}{2}\right) - \frac{1}{2}$. [1]

The set S consists of all real numbers greater than $-\frac{1}{2}$.

(iii) (A) Use the result in part **(ii)** to show that S is closed under the operation $*$.

(B) Show that S, with the operation $*$, is a group. [9]

(iv) Show that S contains no element of order 2. [3]

The group $G = \{0, 1, 2, 4, 5, 6\}$ has binary operation $\circ$ defined by

$x \circ y$ is the remainder when $x * y$ is divided by 7.

(v) Show that $4 \circ 6 = 2$. [2]

The composition table for G is as follows.

$\circ$	0	1	2	4	5	6
0	0	1	2	4	5	6
1	1	4	0	6	2	5
2	2	0	5	1	6	4
4	4	6	1	5	0	2
5	5	2	6	0	4	1
6	6	5	4	2	1	0

(vi) Find the order of each element of G. [3]

(vii) List all the subgroups of G. [3]

MEI FP3 2009

The group $G = \{1, 2, 3, 4, 5, 6\}$ has multiplication modulo 7 as its operation. The group $H = \{1, 5, 7, 11, 13, 17\}$ has multiplication modulo 18 as its operation.

(i) Show that the groups G and H are both cyclic. **[4]**

(ii) List all the proper subgroups of G. **[3]**

(iii) Specify an isomorphism between G and H. **[4]**

The group $S = \{a, b, c, d, e, f\}$ consists of functions with domain $\{1, 2, 3\}$ given by

$a(1) = 2$	$a(2) = 3$	$a(3) = 1$
$b(1) = 3$	$b(2) = 1$	$b(3) = 2$
$c(1) = 1$	$c(2) = 3$	$c(3) = 2$
$d(1) = 3$	$d(2) = 2$	$d(3) = 1$
$e(1) = 1$	$e(2) = 2$	$e(3) = 3$
$f(1) = 2$	$f(2) = 1$	$f(3) = 3$

and the group operation is composition of functions.

(iv) Show that $ad = c$ and find da. **[4]**

(v) Give a reason why S is not isomorphic to G. **[1]**

(vi) Find the order of each element of S. **[4]**

(vii) List all the proper subgroups of S. **[4]**

MEI FP3 2010 - Ignore parts (v) and (viii)

The group $F = \{p, q, r, s, t, u\}$ consists of the six functions defined by

$$p(x) = x \qquad q(x) = 1 - x \qquad r(x) = \frac{1}{x} \qquad s(x) = \frac{x-1}{x} \qquad t(x) = \frac{x}{x-1} \qquad u(x) = \frac{1}{1-x},$$

the binary operation being composition of functions.

(i) Show that st = r and find ts. **[4]**

(ii) Copy and complete the following composition table for F. **[3]**

	p	q	r	s	t	u
p	p	q	r	s	t	u
q	q	p	s	r	u	t
r	r	u	p	t	s	q
s	s	t	q	u	r	p
t	t	s	u			
u	u	r	t			

(iii) Give the inverse of each element of F. **[3]**

(iv) List all the subgroups of F. **[4]**

The group M consists of $\left\{1, -1, e^{\frac{\pi}{3}j}, e^{-\frac{\pi}{3}j}, e^{\frac{2\pi}{3}j}, e^{-\frac{2\pi}{3}j}\right\}$ with multiplication of complex numbers as its binary operation.

(v) Find the order of each element of M. **[4]**

The group G consists of the positive integers between 1 and 18 inclusive, under multiplication modulo 19.

(vi) Show that G is a cyclic group which can be generated by the element 2. **[3]**

(vii) Explain why G has no subgroup which is isomorphic to F. **[1]**

(viii) Find a subgroup of G which is isomorphic to M. **[2]**

MEI FP3 2011 – Ignore part (iii)

(i) Show that the set $G = \{1, 3, 4, 5, 9\}$, under the binary operation of multiplication modulo 11, is a group. You may assume associativity. **[6]**

(ii) Explain why any two groups of order 5 must be isomorphic to each other. **[3]**

The set $H = \left\{1, e^{\frac{2}{5}\pi j}, e^{\frac{4}{5}\pi j}, e^{\frac{6}{5}\pi j}, e^{\frac{8}{5}\pi j}\right\}$ is a group under the binary operation of multiplication of complex numbers.

(iii) Specify an isomorphism between the groups G and H. **[3]**

The set K consists of the 25 ordered pairs (x, y), where x and y are elements of G. The set K is a group under the binary operation defined by

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$$

where the multiplications are carried out modulo 11; for example, $(3, 5)(4, 4) = (1, 9)$.

(iv) Write down the identity element of K, and find the inverse of the element $(9, 3)$. **[2]**

(v) Explain why $(x, y)^5 = (1, 1)$ for every element (x, y) in K. **[3]**

(vi) Deduce that all the elements of K, except for one, have order 5. State which is the exceptional element. **[3]**

(vii) A subgroup of K has order 5 and contains the element $(9, 3)$. List the elements of this subgroup. **[2]**

(viii) Determine how many subgroups of K there are with order 5. **[2]**

MEI FP3 2012

(i) Show that the set $P = \{1, 5, 7, 11\}$, under the binary operation of multiplication modulo 12, is a group. You may assume associativity. **[4]**

A group Q has identity element e. The result of applying the binary operation of Q to elements x and y is written xy, and the inverse of x is written x^{-1}.

(ii) Verify that the inverse of xy is $y^{-1}x^{-1}$. **[2]**

Three elements a, b and c of Q all have order 2, and $ab = c$.

(iii) By considering the inverse of c, or otherwise, show that $ba = c$. **[2]**

(iv) Show that $bc = a$ and $ac = b$. Find cb and ca. **[4]**

(v) Complete the composition table for $R = \{e, a, b, c\}$. Hence show that R is a subgroup of Q and that R is isomorphic to P. **[4]**

The group T of symmetries of a square contains four reflections A, B, C, D, the identity transformation E and three rotations F, G, H. The binary operation is composition of transformations. The composition table for T is given below.

	A	B	C	D	E	F	G	H
A	E	G	H	F	A	D	B	C
B	G	E	F	H	B	C	A	D
C	F	H	E	G	C	A	D	B
D	H	F	G	E	D	B	C	A
E	A	B	C	D	E	F	G	H
F	C	D	B	A	F	G	H	E
G	B	A	D	C	G	H	E	F
H	D	C	A	B	H	E	F	G

(vi) Find the order of each element of T. **[3]**

(vii) List all the proper subgroups of T. **[5]**

MEI FP3 2013

(a) The composition table for a group G of order 8 is given below.

	a	b	c	d	e	f	g	h
a	c	e	b	f	a	h	d	g
b	e	c	a	g	b	d	h	f
c	b	a	e	h	c	g	f	d
d	f	g	h	a	d	c	e	b
e	a	b	c	d	e	f	g	h
f	h	d	g	c	f	b	a	e
g	d	h	f	e	g	a	b	c
h	g	f	d	b	h	e	c	a

(i) State which is the identity element, and give the inverse of each element of G. **[3]**

(ii) Show that G is cyclic. **[4]**

(iii) Specify an isomorphism between G and the group H consisting of $\{0, 2, 4, 6, 8, 10, 12, 14\}$ under addition modulo 16. **[3]**

(iv) Show that G is not isomorphic to the group of symmetries of a square. **[2]**

(b) The set S consists of the functions $f_n(x) = \dfrac{x}{1 + nx}$, for all integers n, and the binary operation is composition of functions.

(i) Show that $f_m f_n = f_{m+n}$. **[2]**

(ii) Hence show that the binary operation is associative. **[2]**

(iii) Prove that S is a group. **[6]**

(iv) Describe one subgroup of S which contains more than one element, but which is not the whole of S. **[2]**

MEI FP3 2014

The twelve distinct elements of an abelian multiplicative group G are

$$e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b$$

where e is the identity element, $a^6 = e$ and $b^2 = e$.

(i) Show that the element a^2b has order 6. **[3]**

(ii) Show that $\{e, a^3, b, a^3b\}$ is a subgroup of G. **[3]**

(iii) List all the cyclic subgroups of G. **[6]**

You are given that the set

$$H = \{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 77, 79, 83, 89\}$$

with binary operation multiplication modulo 90 is a group.

(iv) Determine the order of each of the elements 11, 17 and 19. **[4]**

(v) Give a cyclic subgroup of H with order 4. **[2]**

(vi) By identifying possible values for the elements a and b above, or otherwise, give one example of each of the following:

(A) a non-cyclic subgroup of H with order 12, **[3]**

(B) a non-cyclic subgroup of H with order 4. **[3]**

MEI FP3 2016

(a) The elements of the set $P = \{1, 3, 9, 11\}$ are combined under the binary operation, *, defined as multiplication modulo 16.

(i) Demonstrate associativity for the elements 3, 9, 11 in that order.

Assuming associativity holds in general, show that P forms a group under the binary operation *. **[6]**

(ii) Write down the order of each element. **[2]**

(iii) Write down all subgroups of P. **[1]**

(iv) Show that the group in part **(i)** is cyclic. **[1]**

(b) Now consider a group of order 4 containing the identity element e and the two distinct elements, a and b, where $a^2 = b^2 = e$. Construct the composition table. Show that the group is non-cyclic. **[4]**

(c) Now consider the four matrices **I**, **X**, **Y** and **Z** where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{X} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{Z} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The group G consists of the set $\{\mathbf{I}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}\}$ with binary operation matrix multiplication. Determine which of the groups in parts **(a)** and **(b)** is isomorphic to G, and specify the isomorphism. **[6]**

(d) The distinct elements $\{p, q, r, s\}$ are combined under the binary operation $\circ$. You are given that $p \circ q = r$ and $q \circ p = s$.

By reference to the group axioms, prove that $\{p, q, r, s\}$ is not a group under $\circ$. **[4]**