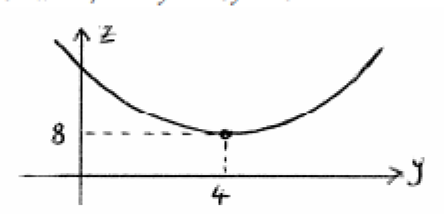
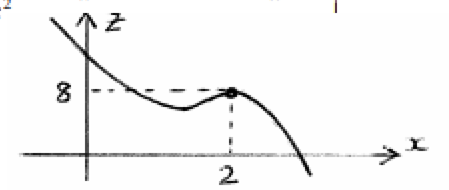
Multi-Variable Calculus Past Paper Question Pack – Mark Schemes

MEI FP3 June 2006

2 (i)	Normal vector is $\begin{pmatrix} 2x-4y \\ -4x+6y \\ -4z \end{pmatrix}$	M1 A1 A1 A1 **4**	Partial differentiation Condone $\mathbf{r}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}+\lambda\begin{pmatrix} 2x-4y \\ -4x+6y \\ -4z \end{pmatrix}$ *For 4 marks the normal must appear as a vector (isw)*
(ii)	At Q normal vector is $\begin{pmatrix} 18 \\ -44 \\ -4 \end{pmatrix}$ Tangent plane is $18x-44y-4z=306-176-4=126$ $9x-22y-2z=63$	M1 M1 M1 A1 **4**	 For $18x-44y-4z$ *Dependent on previous M1* Using Q to find constant Accept any correct form
(iv)	Normal parallel to *z*-axis requires $2x-4y=0$ and $-4x+6y=0$ $x=y=0$; then $-2z^2-63=0$ No solutions; hence no such points	M1A1 ft M1 A1 (ag) **4**	 Correctly shown
	OR $2x-4y=-4x+6y$, so $y=\frac{3}{5}x$ $-\frac{8}{25}x^2-2z^2-63=0$, hence no points M2A2		Similarly if only $2x-4y=0$ used
(v)	$2x-4y=5\lambda$ $-4x+6y=-6\lambda$ $-4z=2\lambda$ $x=-\frac{3}{2}\lambda,\ y=-2\lambda,\ z=-\frac{1}{2}\lambda$ Substituting into equation of surface $\frac{9}{4}\lambda^2-12\lambda^2+12\lambda^2-\frac{1}{2}\lambda^2-63=0$ $\lambda=\pm 6$	M1A1 ft M1 M1 M1 M1	 Obtaining *x*, *y*, *z* in terms of λ or $x=3z,\ y=4z$ Obtaining a value of λ (or equivalent)

MEI FP3 June 2007

(i)	$\frac{\partial z}{\partial x} = y^2 - 8xy - 6x^2 + 54x - 36$	B2	Give B1 for 3 terms correct
	$\frac{\partial z}{\partial y} = 2xy - 4x^2$	B1	
		3	
(ii)	At stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$	M1	
	When $x = 0$, $y^2 - 36 = 0$	M1	
	$y = \pm 6$; points $(0,\ 6,\ 20)$ and $(0,\ -6,\ 20)$	A1A1	If A0, give A1 for $y = \pm 6$
	When $y = 2x$, $4x^2 - 16x^2 - 6x^2 + 54x - 36 = 0$	M1	
	$-18x^2 + 54x - 36 = 0$ $x = 1,\ 2$	M1A1	or $y = 2,\ 4$
	Points $(1,\ 2,\ 5)$ and $(2,\ 4,\ 8)$	A1	*A0 if any extra points given*
		8	
(iii)	When $x = 2$, $z = 2y^2 - 16y + 40$		
		B1 B1	'Upright' parabola $(2,\ 4,\ 8)$ identified as a minimum (in the first quadrant)
	When $y = 4$, $z = -2x^3 + 11x^2 - 20x + 20$		
	$\left(\frac{d^2 z}{dx^2} = -12x + 22 = -2 \text{ when } x = 2 \right)$		
		B1 B1 B1	'Negative cubic' curve $(2,\ 4,\ 8)$ identified as a stationary point Fully correct (unambiguous minimum and maximum)
	The point is a minimum on one section and a maximum on the other; so it is neither a maximum nor a minimum	B1	
		6	
(iv)	Require $\frac{\partial z}{\partial x} = -36$ and $\frac{\partial z}{\partial y} = 0$	M1	$\frac{\partial z}{\partial x} = 36$ can earn all M marks
	When $x = 0$, $y^2 - 36 = -36$	M1	
	$y = 0$; point $(0,\ 0,\ 20)$	A1	
	When $y = 2x$, $4x^2 - 16x^2 - 6x^2 + 54x - 36 = -36$	M1	
	$-18x^2 + 54x = 0$ $x = 0,\ 3$	M1	Solving to obtain x (or y) *or stating 'no roots' if appropriate (e.g. when +36 has been used)*
	$x = 0$ gives $(0,\ 0,\ 20)$ same as above	A1	
	$x = 3$ gives $(3,\ 6,\ -7)$	A1	
		7	

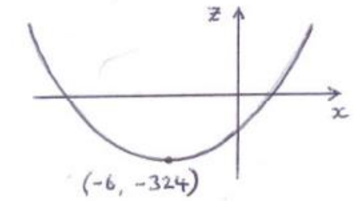
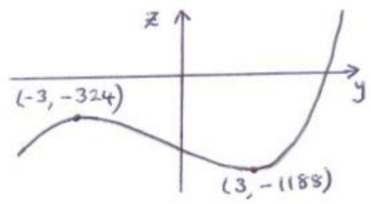
MEI FP3 June 2008

(i)	$\frac{\partial g}{\partial x} = 6z - 2(x+2y+3z) = -2x-4y$ $\frac{\partial g}{\partial y} = -4(x+2y+3z)$ $\frac{\partial g}{\partial z} = 6x - 6(x+2y+3z) = -12y-18z$	M1 A1 A1 A1 **4**	Partial differentiation *Any correct form, ISW*
(ii)	At P, $\frac{\partial g}{\partial x} = 16,\ \frac{\partial g}{\partial y} = -4,\ \frac{\partial g}{\partial z} = 36$ Normal line is $\mathbf{r} = \begin{pmatrix} 7 \\ -7.5 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -1 \\ 9 \end{pmatrix}$	M1 A1 A1 ft **3**	Evaluating partial derivatives at P All correct *Condone omission of '* **r =** *'*
(iv)	Require $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$ $-2x-4y=0$ and $x+2y+3z=0$ $x+2y=0$ and $z=0$ $g(x, y, z) = 0 - 0^2 = 0 \neq 125$ Hence there is no such point on *S*	M1 M1 M1 A1 **4**	 Useful manipulation using both eqns Showing there is no such point on *S* Fully correct proof
(v)	Require $\frac{\partial g}{\partial z} = 0$ and $\frac{\partial g}{\partial y} = 5\frac{\partial g}{\partial x}$ $-4x-8y-12z = 5(-2x-4y)$	M1 M1 M1	 Implied by $\frac{\partial g}{\partial x} = \lambda,\ \frac{\partial g}{\partial y} = 5\lambda$ *This M1 can be awarded for* $-2x-4y=1$ and $-4x-8y-12z=5$

MEI FP3 June 2009

2 (i)	$\frac{\partial z}{\partial x} = 3(x+y)^3 + 9x(x+y)^2 - 6x^2 + 24$ $\frac{\partial z}{\partial y} = 9x(x+y)^2$	M1 A2 A1 **4**	Partial differentiation Give A1 if just one minor error
(ii)	At stationary points, $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ $9x(x+y)^2 = 0 \Rightarrow x = 0$ or $y = -x$ If $x = 0$ then $3y^3 + 24 = 0$ $y = -2$; one stationary point is $(0,\ -2,\ 0)$ If $y = -x$ then $-6x^2 + 24 = 0$ $x = \pm 2$; stationary points are $(2,\ -2,\ 32)$ and $(-2,\ 2,\ -32)$	M1 M1 A1A1 M1 A1 A1 **7**	If A0A0, give A1 for $x = \pm 2$
(iii)	At P $(1,\ -2,\ 19)$, $\frac{\partial z}{\partial x} = 24$, $\frac{\partial z}{\partial y} = 9$ Normal line is $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 19 \end{pmatrix} + \lambda \begin{pmatrix} 24 \\ 9 \\ -1 \end{pmatrix}$	B1 M1 A1 ft **3**	For normal vector (allow sign error) Condone omission of '**r** ='
(v)	$\frac{\partial z}{\partial x} = 27$ and $\frac{\partial z}{\partial y} = 0$ $9x(x+y)^2 = 0 \Rightarrow x = 0$ or $y = -x$ If $x = 0$ then $3y^3 + 24 = 27$ $y = 1,\ z = 0$; point is $(0,\ 1,\ 0)$ $d = 0$ If $y = -x$ then $-6x^2 + 24 = 27$ $x^2 = -\frac{1}{2}$; there are no other points	M1 M1 A1 A1 M1 A1 **6**	(Allow M1 for $\frac{\partial z}{\partial x} = -27$)

MEI FP3 June 2011

(i)	When $y=-3$, $z=3x^2+36x-216$	B1	
		B1	Correct shape (parabola) and position
		B1	For $(-6,-324)$
	When $x=-6$, $z=8y^3-216y-756$	B1	
		B1	Correct shape and position
		B1	For $(-3,-324)$
		B1	For $(3,-1188)$
		7	*If B0B0 then give B1 for* $x=\pm 3$
(ii)	$(-6,-3,-324)$ is a SP on both sections; hence it is a SP on S	B1	
	Saddle point	B1	
		2	
(iii)	$\frac{\partial z}{\partial x}=-12xy-30x+36$, $\frac{\partial z}{\partial y}=24y^2-6x^2$	B1B1	
	At a SP, $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=0$	M1	
	$24y^2-6x^2=0 \Rightarrow y=\pm\frac{1}{2}x$		
	$y=\frac{1}{2}x \Rightarrow -6x^2-30x+36=0$	M1	
	$\Rightarrow x=-6,\ 1;$ SPs are $(-6,-3,-324)$		
	$(1,\ 0.5,\ 19)$	A1	
	$y=-\frac{1}{2}x \Rightarrow 6x^2-30x+36=0$	M1	
	$\Rightarrow x=2,\ 3;$ SPs are $(2,-1,\ 28)$	A1	
	$(3,-1.5,\ 27)$	A1	
		8	
(iv)	$\frac{\partial z}{\partial x}=120$ and $\frac{\partial z}{\partial y}=0$	M1	(Allow M1 for $\frac{\partial z}{\partial x}=-120$)
	$y=\frac{1}{2}x \Rightarrow -6x^2-30x-84=0;\ D=30^2-4\times 6\times 84$	M1	
	$D\ (=-1116)<0$; so this has no roots	A1	
	$y=-\frac{1}{2}x \Rightarrow 6x^2-30x-84=0 \Rightarrow x=7,-2$	M1	Obtaining at least one value of x
		M1	Obtaining a value of k
	When $x=7$, $y=-3.5$, $z=203$; so $k=637$	A1	
	When $x=-2$, $y=1$, $z=-148$; so $k=-92$	A1	
		7	

MEI FP3 June 2012

(i)		$\frac{\partial g}{\partial x} = 2x + 2z$ $\frac{\partial g}{\partial y} = 4y + 2z$ $\frac{\partial g}{\partial z} = -2z + 2x + 2y + 4$	B1 B1 B1 **[3]**		
(ii)		At P, $\frac{\partial g}{\partial x} = -2,\ \frac{\partial g}{\partial y} = -2,\ \frac{\partial g}{\partial z} = -4$ Normal line is $\mathbf{r} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	B1 M1 A1 **[3]**	For direction of normal line FT	*Condone omission of* $\mathbf{r} =$
(iv)		Require $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = 0$ $x = -z,\ \ y = -\frac{1}{2}z$ $z^2 + \frac{1}{2}z^2 - z^2 - 2z^2 - z^2 + 4z - 3 = 0$ $5z^2 - 8z + 6 = 0$ Discriminant is $64 - 120 = -56 < 0$ Hence there are no such points	M1 M1 A1 M1 E1 **[5]**	Obtaining equation in one variable Or $5x^2 + 8x + 6 = 0$ or $10y^2 + 8y + 3 = 0$ or $\lambda^2 + 14 = 0$ *Dependent on quadratic with negative discriminant* Correctly shown	($\lambda = \frac{\partial g}{\partial z}$)
(v)		Require $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial y} = \frac{\partial g}{\partial z}$ $(= \lambda)$ $2x + 2z = 4y + 2z = -2z + 2x + 2y + 4$ $x = 2y,\ \ y = 2z - 2$ $(4z-4)^2 + 2(2z-2)^2 - z^2 + \ldots + 4z - 3 = 0$ $5z^2 - 8z + 3 = 0$ Points $(0,\ 0,\ 1)$ and $(-1.6,\ -0.8,\ 0.6)$ $k = 0 + 0 + 1$ or $k = -1.6 - 0.8 + 0.6$ $k = 1,\ \ -1.8$	M1 A1 M1 A1 M1 M1 A1A1 **[8]**	*Allow M1 if* $\lambda = 1$ FT Obtaining equation in one variable Or $5x^2 + 8x = 0$ or $5y^2 + 4y = 0$ Obtaining at least one point Obtaining a value of k	Or $\lambda^2 - 4 = 0$ *Implies previous M1 if values of x, y, z not seen*

MEI FP3 June 2013

(i)		$\frac{\partial z}{\partial x} = 6x^2 + 6x + 12y$	B1		
		$\frac{\partial z}{\partial y} = 6y^2 + 6y + 12x$	B1		
		If $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$, $6x^2 + 6x + 12y = 6y^2 + 6y + 12x$			
		$x^2 - y^2 - x + y = 0$			
		$(x-y)(x+y-1) = 0$	M1	Identifying factor $(x-y)$	*SC* If M0, then give
		$y = x$ or $y = 1-x$	E1E1		B1 for verifying $y = x$ B1 for verifying $y = 1-x$
			[5]		
(ii)		$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$	M1		Can be implied
		If $y = x$ then $6x^2 + 6x + 12x = 0$	M1	Obtaining quadratic in x (or y)	Or quartic, and factorising as x(linear)(quadratic)
		$x = 0, \ -3$	M1	Obtaining a non-zero value of x	
		Stationary points $(0, \ 0, \ 0)$ and $(-3, \ -3, \ 54)$	B1A1	Condone $(0, \ 0)$ for B1	
		If $y = 1-x$ then $6x^2 + 6x + 12(1-x) = 0$			
		$x^2 - x + 2 = 0$	M1	Obtaining quadratic with no real roots	
		Which has no real roots ($D = -7 < 0$)	A1	Correctly shown	Just stating 'No real roots' M1A0
			[7]		
(iii)		At P, $\frac{\partial z}{\partial x} = \frac{21}{2}$, $\frac{\partial z}{\partial y} = \frac{21}{2}$	M1 A1	Substituting into $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$	Correct value, or substitution seen
		$\delta z \approx \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y$	M1		
		$w \approx \frac{21}{2}h + \frac{21}{2}h$	A1 ft		
		$h \approx \frac{w}{21}$	A1		
(iv)		$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 24$	M1	Allow sign error	24λ is M0 unless $\lambda = \pm 1$ appears later
		If $y = x$ then $6x^2 + 6x + 12x = 24$	M1	Obtaining quadratic in x (or y)	Or quartic, and one linear factor
		$x = 1, \ -4$			
		Points $(1, \ 1, \ 22)$ and $(-4, \ -4, \ 32)$	A1A1	If neither correct, give A1 for $x = 1, \ -4$	
		If $y = 1-x$ then $6x^2 + 6x + 12(1-x) = 24$	M1	Obtaining quadratic in x (or y)	Or third linear factor of quartic
		$x = 2, \ -1$			
		Points $(2, \ -1, \ 5)$ and $(-1, \ 2, \ 5)$	A1A1	If neither correct, give A1 for $x = 2, \ -1$	
			[7]		

MEI FP3 June 2014

(i)		$\frac{\partial g}{\partial x} = 2x + 6z - 4y$ $\frac{\partial g}{\partial y} = 6y + 2z - 4x$ $\frac{\partial g}{\partial z} = 4z + 2y + 6x$	B1 B1 B1 **[3]**		
(ii)		At P, $\frac{\partial g}{\partial x} = -32,\ \frac{\partial g}{\partial y} = 24,\ \frac{\partial g}{\partial z} = 16$ Normal line is $\mathbf{r} = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -4 \\ 3 \\ 2 \end{pmatrix}$	B1 M1 A1 **[3]**	 Direction of normal line FT	 Condone omission of ' $\mathbf{r}$ = '
(iv)		Require $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial z} = 0$ $2x + 6z - 4y = 0$ and $4z + 2y + 6x = 0$ $y = -x,\ \ z = -x$ $x^2 + 3x^2 + 2x^2 + 2x^2 - 6x^2 + 4x^2 - 24 = 0$ $6x^2 - 24 = 0$ Points $(2,\ -2,\ -2)$ and $(-2,\ 2,\ 2)$	M1 M1 M1 A1 A1A1 **[6]**	 For (e.g.) y and z as multiples of x Quadratic in one variable In simplified form If neither point correct, give A1 for any four correct coordinates	
(v)		$\begin{pmatrix} 2x + 6z - 4y \\ 6y + 2z - 4x \\ 4z + 2y + 6x \end{pmatrix} = \lambda \begin{pmatrix} 10 \\ -1 \\ 2 \end{pmatrix}$ $y = 3x,\ z = -5x$	M1 A1 FT M1	Allow M1 even if $\lambda = 1$ For (e.g.) y and z as multiples of x	 Or $x = -\frac{1}{4}\lambda,\ y = -\frac{3}{4}\lambda,\ z = \frac{5}{4}\lambda$