Matrices Past Paper Question Pack

MEI FP2 Jan 2006

The matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ -2 & -3 & 6 \\ 2 & 2 & -4 \end{pmatrix}$.

(i) Show that the characteristic equation for $\mathbf{M}$ is $\lambda^3 + 6\lambda^2 - 9\lambda - 14 = 0$. [3]

(ii) Show that -1 is an eigenvalue of $\mathbf{M}$, and find the other two eigenvalues. [3]

(iii) Find an eigenvector corresponding to the eigenvalue -1. [3]

(iv) Verify that $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 3 \\ -2 \end{pmatrix}$ are eigenvectors of $\mathbf{M}$. [3]

(v) Write down a matrix $\mathbf{P}$, and a diagonal matrix $\mathbf{D}$, such that $\mathbf{M}^3 = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$. [3]

(vi) Use the Cayley-Hamilton theorem to express $\mathbf{M}^{-1}$ in the form $a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}$. [3]

MEI FP2 Jan 2007

Let $\mathbf{P} = \begin{pmatrix} 4 & 2 & k \\ 1 & 1 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ (where $k \neq 4$) and $\mathbf{M} = \begin{pmatrix} 2 & -2 & -6 \\ -1 & 3 & 1 \\ 1 & -2 & -2 \end{pmatrix}$.

(i) Find $\mathbf{P}^{-1}$ in terms of k, and show that, when $k = 2$, $\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 & 4 \\ 4 & -6 & -10 \\ -1 & 2 & 2 \end{pmatrix}$. [6]

(ii) Verify that $\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ are eigenvectors of $\mathbf{M}$, and find the corresponding eigenvalues. [4]

(iii) Show that $\mathbf{M}^n = \begin{pmatrix} 4 & -6 & -10 \\ 2 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix} + 2^{n-1}\begin{pmatrix} -2 & 4 & 4 \\ -3 & 6 & 6 \\ 1 & -2 & -2 \end{pmatrix}$. [8]

MEI FP2 June 2007

Let $\mathbf{M} = \begin{pmatrix} 3 & 5 & 2 \\ 5 & 3 & -2 \\ 2 & -2 & -4 \end{pmatrix}$.

(i) Show that the characteristic equation for **M** is $\lambda^3 - 2\lambda^2 - 48\lambda = 0$. [4]

You are given that $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector of **M** corresponding to the eigenvalue 0.

(ii) Find the other two eigenvalues of **M**, and corresponding eigenvectors. [8]

(iii) Write down a matrix **P**, and a diagonal matrix **D**, such that $\mathbf{P}^{-1}\mathbf{M}^2\mathbf{P} = \mathbf{D}$. [3]

(iv) Use the Cayley-Hamilton theorem to find integers a and b such that $\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M}$. [3]

MEI FP2 Jan 2008

You are given the matrix $\mathbf{M} = \begin{pmatrix} 7 & 3 \\ -4 & -1 \end{pmatrix}$.

(i) Find the eigenvalues, and corresponding eigenvectors, of the matrix **M**. [8]

(ii) Write down a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P} = \mathbf{D}$. [2]

(iii) Given that $\mathbf{M}^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $a = -\frac{1}{2} + \frac{3}{2} \times 5^n$, and find similar expressions for b, c and d.
[8]

MEI FP2 June 2008

(i) Given the matrix $\mathbf{Q} = \begin{pmatrix} 2 & -1 & k \\ 1 & 0 & 1 \\ 3 & 1 & 2 \end{pmatrix}$ (where $k \neq 3$), find $\mathbf{Q}^{-1}$ in terms of k.

Show that, when $k = 4$, $\mathbf{Q}^{-1} = \begin{pmatrix} -1 & 6 & -1 \\ 1 & -8 & 2 \\ 1 & -5 & 1 \end{pmatrix}$. [6]

The matrix $\mathbf{M}$ has eigenvectors $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$, with corresponding eigenvalues 1, −1 and 3 respectively.

(ii) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P} = \mathbf{D}$, and hence find the matrix $\mathbf{M}$. [7]

(iii) Write down the characteristic equation for $\mathbf{M}$, and use the Cayley-Hamilton theorem to find integers a, b and c such that $\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}$. [5]

MEI FP2 Jan 09

(i) Find the eigenvalues and corresponding eigenvectors for the matrix **M** where

$$\mathbf{M} = \begin{pmatrix} 0.2 & 0.8 \\ 0.3 & 0.7 \end{pmatrix}.$$ **[6]**

(ii) Give a matrix **Q** and a diagonal matrix **D** such that $\mathbf{M} = \mathbf{QDQ}^{-1}$. **[3]**

MEI FP2 Jan 11

(i) Show that the characteristic equation of the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -4 & 5 \\ 2 & 3 & -2 \\ -1 & 4 & 1 \end{pmatrix}$$

is $\lambda^3 - 5\lambda^2 + 28\lambda - 66 = 0$. **[4]**

(ii) Show that $\lambda = 3$ is an eigenvalue of $\mathbf{M}$, and determine whether or not $\mathbf{M}$ has any other real eigenvalues. **[4]**

(iii) Find an eigenvector, $\mathbf{v}$, of unit length corresponding to $\lambda = 3$.

State the magnitude of the vector $\mathbf{M}^n\mathbf{v}$, where n is an integer. **[5]**

(iv) Using the Cayley-Hamilton theorem, obtain an equation for $\mathbf{M}^{-1}$ in terms of $\mathbf{M}^2$, $\mathbf{M}$ and $\mathbf{I}$. **[3]**

MEI FP2 June 2011

(i) Find the value of k for which the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & -1 & k \\ 5 & 4 & 6 \\ 3 & 2 & 4 \end{pmatrix}$$

does not have an inverse.

Assuming that k does not take this value, find the inverse of $\mathbf{M}$ in terms of k. **[7]**

(ii) In the case $k = 3$, evaluate

$$\mathbf{M}\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}.$$ **[2]**

(iii) State the significance of what you have found in part **(ii)**. **[2]**

MEI FP2 Jan 2012

(i) Show that the characteristic equation of the matrix

$$\mathbf{M} = \begin{pmatrix} 3 & -1 & 2 \\ -4 & 3 & 2 \\ 2 & 1 & -1 \end{pmatrix}$$

is $\lambda^3 - 5\lambda^2 - 7\lambda + 35 = 0$. **[4]**

(ii) Show that $\lambda = 5$ is an eigenvalue of $\mathbf{M}$, and find its other eigenvalues. **[4]**

(iii) Find an eigenvector, $\mathbf{v}$, of unit length corresponding to $\lambda = 5$.

State the magnitudes and directions of the vectors $\mathbf{M}^2\mathbf{v}$ and $\mathbf{M}^{-1}\mathbf{v}$. **[6]**

(iv) Use the Cayley-Hamilton theorem to find the constants a, b, c such that

$$\mathbf{M}^4 = a\mathbf{M}^2 + b\mathbf{M} + c\mathbf{I}.$$ **[4]**

MEI FP2 Jan 2013

You are given the matrix $\mathbf{M} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

(i) Show that the characteristic equation of $\mathbf{M}$ is

$$\lambda^3 - 13\lambda + 12 = 0.$$ **[3]**

(ii) Find the eigenvalues and corresponding eigenvectors of $\mathbf{M}$. **[12]**

(iii) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that

$$\mathbf{M}^n = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}.$$

(You are not required to calculate $\mathbf{P}^{-1}$.) **[3]**

MEI FP2 June 2015

This question concerns the matrix $\mathbf{M}$ where $\mathbf{M} = \begin{pmatrix} 5 & -1 & 3 \\ 4 & -3 & -2 \\ 2 & 1 & 4 \end{pmatrix}$.

(i) Obtain the characteristic equation of $\mathbf{M}$.

Find the eigenvalues of $\mathbf{M}$. **[7]**

These eigenvalues are denoted by λ_1, λ_2, λ_3, where $\lambda_1 < \lambda_2 < \lambda_3$.

(ii) Verify that an eigenvector corresponding to λ_1 is $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ and that an eigenvector corresponding to λ_2 is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$. Find an eigenvector of the form $\begin{pmatrix} a \\ 1 \\ c \end{pmatrix}$ corresponding to λ_3. **[5]**

(iii) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{M} = \mathbf{PDP}^{-1}$. (You are not required to calculate $\mathbf{P}^{-1}$.)

Hence write down an expression for $\mathbf{M}^4$ in terms of $\mathbf{P}$ and a diagonal matrix. You should give the elements of the diagonal matrix explicitly. **[3]**

(iv) Use the Cayley-Hamilton theorem to obtain an expression for $\mathbf{M}^4$ as a linear combination of $\mathbf{M}$ and $\mathbf{M}^2$. **[3]**