Matrices 2 Past Paper Question Pack – Mark Schemes

MEI P6 June 02

1 (i)	Characteristic equation is $(k-\lambda)(2-\lambda)-0=0$ Eigenvalues are k and 2 For $\lambda = k$, $\begin{pmatrix} k & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = k\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} kx+3y=kx \\ 2y=ky \end{matrix}$ $\Rightarrow y=0$ so an eigenvector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ For $\lambda = 2$, $\begin{pmatrix} k & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{matrix} kx+3y=2x \\ 2y=2y \end{matrix}$ $\Rightarrow y = \frac{1}{3}(2-k)x$ so an eigenvector is $\begin{pmatrix} 3 \\ 2-k \end{pmatrix}$	M1 A1A1 M1 A1 M1 A1 7	
(ii)	$\mathbf{P} = \begin{pmatrix} 1 & 3 \\ 0 & 2-k \end{pmatrix}$	B2 ft 2	
(iii)	$\mathbf{P}^{-1}\mathbf{MP} = \mathbf{D} = \begin{pmatrix} k & 0 \\ 0 & 2 \end{pmatrix}$ $\mathbf{D}^n = \begin{pmatrix} k^n & 0 \\ 0 & 2^n \end{pmatrix}$ $\mathbf{P}^{-1} = \frac{1}{2-k}\begin{pmatrix} 2-k & -3 \\ 0 & 1 \end{pmatrix}$ $\mathbf{M}^n = \mathbf{PD}^n\mathbf{P}^{-1}$ $= \frac{1}{2-k}\begin{pmatrix} 1 & 3 \\ 0 & 2-k \end{pmatrix}\begin{pmatrix} k^n & 0 \\ 0 & 2^n \end{pmatrix}\begin{pmatrix} 2-k & -3 \\ 0 & 1 \end{pmatrix}$ $= \frac{1}{2-k}\begin{pmatrix} k^n & 3(2^n) \\ 0 & (2-k)2^n \end{pmatrix}\begin{pmatrix} 2-k & -3 \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} k^n & \frac{3(2^n-k^n)}{2-k} \\ 0 & 2^n \end{pmatrix}$	B1 ft B1 ft B1B1 ft M1 A1 cao A1 cao 7	 Give even if order is wrong Or $\frac{1}{2-k}\begin{pmatrix} 1 & 3 \\ 0 & 2-k \end{pmatrix}\begin{pmatrix} (2-k)k^n & -3k^n \\ 0 & 2^n \end{pmatrix}$
(iv)	Characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$ By Cayley-Hamilton theorem, $\mathbf{M}^2 - 3\mathbf{M} + 2\mathbf{I} = \mathbf{O}$ Hence $\mathbf{M}^9 - 3\mathbf{M}^8 + 2\mathbf{M}^7 = \mathbf{O}$ i.e. $\mathbf{M}^9 = 3\mathbf{M}^8 - 2\mathbf{M}^7$ i.e. $p=3,\ q=-2$	B1 ft M1 M1 A1 cao 4	 Applying CH theorem Multiplying by $\mathbf{M}^7$

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(a)(i)	$\mathbf{P}_k^{-1} = \frac{1}{4k}\begin{pmatrix} -7 & 28+3k & -35-2k \\ -3 & 12-k & -15+2k \\ 4 & -16 & 20 \end{pmatrix}$	M1A1 M1 A1 M1 A1 **6**	Calculation of determinant Finding at least 3 cofactors 6 signed cofactors correct Fully correct method for inverse Inverse correct
(ii)	$\mathbf{P}_1^{-1}\mathbf{A}\mathbf{P}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	M1	
	$\mathbf{A} = \mathbf{P}_1\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{P}_1^{-1}$	A1	
	$= \begin{pmatrix} 3 & -7 & 1 \\ 2 & 2 & 5 \\ 1 & 3 & 4 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\frac{1}{4}\begin{pmatrix} -7 & 31 & -37 \\ -3 & 11 & -13 \\ 4 & -16 & 20 \end{pmatrix}$	M1 A1 ft	Substituting $k=1$ in $\mathbf{P}^{-1}$ 3 numerical matrices in correct order
	$= \begin{pmatrix} 10.5 & -42.5 & 50.5 \\ 2 & -10 & 12 \\ -0.5 & 0.5 & -0.5 \end{pmatrix}$	A1 **5**	

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1 (i)	$\begin{pmatrix} 2 & 6 & 8 \\ 2 & 3 & 5 \\ -6 & 8 & 2 \end{pmatrix}\begin{pmatrix} 19 \\ 13 \\ -5 \end{pmatrix}=\begin{pmatrix} 76 \\ 52 \\ -20 \end{pmatrix}$ $=4\begin{pmatrix} 19 \\ 13 \\ -5 \end{pmatrix}$ hence it is an eigenvector with eigenvalue 4	M1A1 A1 A1 **4**	If done as part of (ii), B2 for eigenvalue 4 correctly obtained B2 for $\begin{pmatrix} 19 \\ 13 \\ -5 \end{pmatrix}$ correctly obtained
(ii)	$\det(\mathbf{M}-\lambda\mathbf{I})=-\lambda^3+7\lambda^2-12\lambda$ For eigenvalues, $-\lambda^3+7\lambda^2-12\lambda=0$ $-\lambda(\lambda-3)(\lambda-4)=0$ Other eigenvalues are $\lambda=0,\ 3$ If $\lambda=0$, $2x+6y+8z=0$ $2x+3y+5z=0$ $x=-z,\quad y=-z$ Eigenvector is $\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ If $\lambda=3$, $2x+6y+8z=3x$ $2x+3y+5z=3y$ $x=-\frac{5}{2}z,\quad y=-\frac{7}{4}z$ Eigenvector is $\begin{pmatrix} -10 \\ -7 \\ 4 \end{pmatrix}$	M1A1 M1 A1 M1 A1 M1 A1 **8**	Solving characteristic equation Or $\begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix}\times\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}\left[=\begin{pmatrix} 6 \\ 6 \\ -6 \end{pmatrix}\right]$ Or $\begin{pmatrix} -1 \\ 6 \\ 8 \end{pmatrix}\times\begin{pmatrix} 2 \\ 0 \\ 5 \end{pmatrix}\left[=\begin{pmatrix} 30 \\ 21 \\ -12 \end{pmatrix}\right]$
(iii)	$\mathbf{P}=\begin{pmatrix} 19 & -1 & -10 \\ 13 & -1 & -7 \\ -5 & 1 & 4 \end{pmatrix}$ $\mathbf{D}=\begin{pmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}^4=\begin{pmatrix} 256 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 81 \end{pmatrix}$	B1 ft M1A1 ft **3**	B0 if **P** is clearly singular
(iv)	Characteristic eqn is $-\lambda^3+7\lambda^2-12\lambda=0$ By CHT, $-\mathbf{M}^3+7\mathbf{M}^2-12\mathbf{M}=\mathbf{0}$ $\mathbf{M}^3=7\mathbf{M}^2-12\mathbf{M}$	M1 A1 (ag) **2**	

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(i)	$\det\begin{pmatrix}\frac{1}{2}-\lambda & \frac{1}{2}\\ \frac{2}{3} & \frac{1}{3}-\lambda\end{pmatrix}=0$ $\left(\frac{1}{2}-\lambda\right)\left(\frac{1}{3}-\lambda\right)-\frac{1}{3}=0$	B1	$6\lambda^2-5\lambda-1=0$
	Roots $\lambda=1, -1/6$	B1	
		M1	Using $\mathbf{Mx}=\lambda\mathbf{x}$ or $(\mathbf{M}-\lambda\mathbf{I})\mathbf{x}=0$
	$\lambda=1$: obtain $y=x$ hence eigenvector (e.g.) $\begin{pmatrix}1\\1\end{pmatrix}$	A1	
	$\lambda=-1/6$: obtain $3y=-4x$ hence eigenvector (e.g.) $\begin{pmatrix}3\\-4\end{pmatrix}$	A1	
	$\mathbf{D}=\begin{pmatrix}1 & 0\\ 0 & -\frac{1}{6}\end{pmatrix}$	B1 ft	*For B1B1 the order must be consistent*
	$\mathbf{P}=\begin{pmatrix}1 & 3\\ 1 & -4\end{pmatrix}$	B1 ft	
	$\mathbf{P}^{-1}=-\frac{1}{7}\begin{pmatrix}-4 & -3\\ -1 & 1\end{pmatrix}$	B1 ft **[8]**	The mark for $\mathbf{P}^{-1}$ may be gained in part (ii)

(ii)	$\mathbf{M}^n=\mathbf{PD}^n\mathbf{P}^{-1}$	B1	*Allow matrices written out provided*	*intention is clear*
	$\mathbf{D}^n=\begin{pmatrix}1 & 0\\ 0 & \left(-\frac{1}{6}\right)^n\end{pmatrix}$	B1 ft	"$-\frac{1}{6}^n$" gets B0 unless recovered later	
	Multiply out $\mathbf{PD}^n\mathbf{P}^{-1}$ to obtain	M1		
	$\frac{1}{7}\begin{pmatrix}4+3\left(-\frac{1}{6}\right)^n & 3-3\left(-\frac{1}{6}\right)^n\\ 4-4\left(-\frac{1}{6}\right)^n & 3+4\left(-\frac{1}{6}\right)^n\end{pmatrix}$	A1	All terms required	A0 if not simplified e.g. 1^n
	As n tends to infinity, $\left(-\frac{1}{6}\right)^n$ tends to zero.	M1	*May be implied*	
	$\frac{1}{7}\begin{pmatrix}4 & 3\\ 4 & 3\end{pmatrix}=\begin{pmatrix}\frac{4}{7} & \frac{3}{7}\\ \frac{4}{7} & \frac{3}{7}\end{pmatrix}$	A1 ft **[6]**		
(iii)	$\mathbf{M}^{-1}=\mathbf{PD}^{-1}\mathbf{P}^{-1}$	B1	*Allow matrices written out provided*	*intention is clear*
	$(\mathbf{M}^{-1})^n=\mathbf{PD}^{-n}\mathbf{P}^{-1}$	M1	Or elements of $(\mathbf{M}^{-1})^n$ are the same 'size' as elements of $\mathbf{D}^{-n}$	or M2 for $(\mathbf{M}^{-1})^n$ is the matrix in (ii) with n replaced by $-n$
	$\mathbf{D}^{-1}=\begin{pmatrix}1 & 0\\ 0 & -6\end{pmatrix}$ so $\mathbf{D}^{-n}=\begin{pmatrix}1 & 0\\ 0 & (-6)^n\end{pmatrix}$	M1	Or $\mathbf{D}^{-n}$ contains element $(-6)^n$	
	Hence $(\mathbf{M}^{-1})^n$ does not tend to a limit	A1 **[4]**	*Dependent on M1M1*	

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(i)	det $(\mathbf{Q}-\lambda\mathbf{I}) = 0$ $\Rightarrow$ det $\begin{pmatrix} 3-\lambda & 3 \\ 4 & 7-\lambda \end{pmatrix} = 0$ $\Rightarrow (3-\lambda)(7-\lambda) - 12 = 0$ $\Rightarrow \lambda^2 - 10\lambda + 9 = 0$ so $\lambda = 1, 9$ For $\lambda = 1$, $\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$ $\Rightarrow 2x + 3y = 0$ so eigenvector is $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ o.e For $\lambda = 9$, $\begin{pmatrix} -6 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$ $\Rightarrow 2x - y = 0$ so eigenvector is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ o.e	M1 A1 M1 A1 A1 5	Forming characteristic equation. For either λ, finding eqn in x and y from $(\mathbf{Q}-\lambda\mathbf{I})\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{0}$ o.e.

(b)	(ii)	$\mathbf{P} = \begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}$ $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$	B1ft B1ft 2	 For B2, columns must be consistent
(b)	(iii)	$\mathbf{P}^{-1} = \frac{1}{8}\begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$ $\mathbf{Q}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ $= \frac{1}{8}\begin{pmatrix} 3 & 1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 9^n \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$ $\mathbf{Q}^n = \frac{1}{8}\begin{pmatrix} 3 & 9^n \\ -2 & 2\times 9^n \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 2 & 3 \end{pmatrix}$ $= \frac{1}{8}\begin{pmatrix} 6+2\phi & 3\phi-3 \\ 4\phi-4 & 6\phi+2 \end{pmatrix}$ where $\phi = 9^n$	B1ft M1 DM1 A1 4	(ft provided their **P** has an inverse) Forming product Both matrix products attempted **AG** *By induction: M2A1 for inductive step* *A1 for checking n = 1 and completion*

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(i)	Let $\mathbf{X} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & k \\ k & 1 & 6 \end{pmatrix}$		
	det $\mathbf{X}$ = $3k^2 - 5k + 28$	M1	Allow one error
	$\mathbf{X}^{-1} = \frac{1}{3k^2 - 5k + 28}\begin{pmatrix} 12-k & -16 & 3k-4 \\ k^2+6 & 6-2k & -2-k \\ -1-2k & 3k-1 & 5 \end{pmatrix}$	M1	At least 4 (signed) cofactors correct. **M0** if multiplied by the corresponding element.
		A1	6 (signed) cofactors correct.
		DM1	Transposing and multiplying by 1/det **X**
		A1	cao.
		5	
(ii)	$\mathbf{P} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 2 & 0 \\ 0 & 1 & 6 \end{pmatrix}$	M1 A1	Using the three eigenvectors as columns of a 3x3 matrix. (**M1A0** if columns are in the wrong order)
		2	
(iii)	Using result from **(i)**, with k = 0, $\mathbf{P}^{-1} = \frac{1}{28}\begin{pmatrix} 12 & -16 & -4 \\ 6 & 6 & -2 \\ -1 & -1 & 5 \end{pmatrix}$	M1	Or from scratch with fewer than 3 errors
	$\mathbf{M} = \mathbf{PDP}^{-1}$		
	$\mathbf{PD} = \begin{pmatrix} 3 & 6 & 2 \\ -3 & 4 & 0 \\ 0 & 2 & 6 \end{pmatrix}$	M1 A1 ft	First product. **NB** $\mathbf{DP}^{-1} = \frac{1}{28}\begin{pmatrix} 36 & -48 & -12 \\ 12 & 12 & -4 \\ -1 & -1 & 5 \end{pmatrix}$
	so $\mathbf{M} = \frac{1}{28}\begin{pmatrix} 70 & -14 & -14 \\ -12 & 72 & 4 \\ 6 & 6 & 26 \end{pmatrix}$	DM1 A1	Other product, giving **M** cao *(Correct answer always earns 5 marks)*
		5	
(iv)	Characteristic equation may be expressed as $(\lambda-1)(\lambda-2)(\lambda-3)=0$	M1	**or** expanding $\det(\mathbf{M}-\lambda\mathbf{I})$ (M0 for $(\lambda+1)$ ...)
	i.e. $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$	A1	Alternatively, may be awarded later in terms of **M** and **I** (=0 is not required)
	By the Cayley-Hamilton theorem, **M** must satisfy the characteristic equation, so $\mathbf{M}^3 - 6\mathbf{M}^2 + 11\mathbf{M} - 6\mathbf{I} = \mathbf{0}$	M1	(=0 required; can be implied later) (**I** not required)
	Multiplying by $\mathbf{M}^{-1}$ gives $\mathbf{M}^2 - 6\mathbf{M} + 11\mathbf{I} - 6\mathbf{M}^{-1} = \mathbf{0}$	M1 A1 ft	Cubic expression needed **I** needed (can be recovered later); must be an equation
	$\Rightarrow \mathbf{M}^{-1} = \frac{1}{6}\mathbf{M}^2 - \mathbf{M} + \frac{11}{6}\mathbf{I}$	A1	cao $\left(a = \frac{1}{6}, b = -1, c = \frac{11}{6}\right)$ *Alternatively: **M3** for complete method leading to a value for one of a, b, c **A1A1A1** for answers*
		6	