Matrix algebra

5

Objectives

After completing this chapter you should be able to:

- Find the eigenvalues of a matrix → pages 000–000
- Find the eigenvectors of a matrix → pages 000–000
- Reduce matrices to diagonal form → pages 000–000
- Understand and use the Cayley–Hamilton theorem → pages 000–000

Prior knowledge check

1 Find the determinants of the following matrices.

a $\begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ ← Core Pure Book 1, Chapter 6

2 $\mathbf{A} = \begin{pmatrix} -3 & k \\ 2 & -4 \end{pmatrix}$

Find the value of k such that the matrix **A** is singular. ← Core Pure Book 1, Chapter 6

3 The matrix $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ represents a transformation.

Show that the line with equation $3y + 2x = 0$ is invariant under this transformation. ← Core Pure Book 1, Section 7.2

Reducing a given matrix to diagonal form can help to solve a system of coupled differential equations in a problem involving a closed ecosystem such as a predator–prey model.

← Core Pure Book 2, Chapter 8

→ Exercise 5C, Challenge

5.1 Eigenvalues and eigenvectors

You need to be able to find the **eigenvectors** and **eigenvalues** associated with a square matrix.

- **An eigenvector of a matrix A is a non-zero column vector x which satisfies the equation**

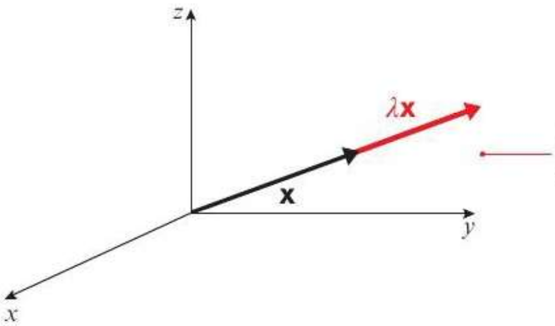
  $$\mathbf{Ax} = \lambda\mathbf{x}$$

  where λ is a scalar.

- **The value of the scalar λ is the eigenvalue of the matrix corresponding to the eigenvector x.**

Notation The word **eigen** is German and means 'particular' or 'special'.

The magnitude of an eigenvector may be changed by the linear transformation represented by the matrix but the direction of the eigenvector is unchanged or **invariant**. The eigenvalue can be interpreted as the magnification factor of the eigenvector under the transformation.



Under the transformation, the eigenvector **x** maps to the vector $\lambda\mathbf{x}$.

- **If x is an eigenvector of a matrix M representing a linear transformation, then the straight line that passes through the origin in the direction of x is an invariant line under that transformation.**

If the corresponding eigenvalue is 1, then every point on this line is an invariant point.

Links The origin is always an **invariant point** under any linear transformation. ← **Core Pure Book 1, Chapter 7**

If **x** is an eigenvector of the matrix **A** then, by definition

$$\mathbf{Ax} = \lambda\mathbf{x} = \lambda\mathbf{Ix}$$

Rearranging,

$$\mathbf{Ax} - \lambda\mathbf{Ix} = (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

As by definition **x** is non-zero, the matrix $(\mathbf{A} - \lambda\mathbf{I})$ is singular and has determinant zero, that is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

Note You can show that if matrix **M** is such that $\mathbf{Mx} = \mathbf{0}$ and **x** is non-zero, then **M** is singular. → **Exercise 5A, Q12**

This means that if you can find a scalar λ that satisfies this equation, then it will be an eigenvalue of **A**.

- **The equation $\det(\mathbf{A} - \lambda\mathbf{I}) = \mathbf{0}$ is called the characteristic equation of A. The solutions to the characteristic equation are the eigenvalues of A.**

In the case of a 2 × 2 matrix, the characteristic equation is quadratic.

Example 1

Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}$.

$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 5 \\ -1 & -4-\lambda \end{pmatrix}$

$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 5 \\ -1 & -4-\lambda \end{vmatrix}$

$= (2-\lambda)(-4-\lambda) - 5 \times (-1)$

$= -8 - 2\lambda + 4\lambda + \lambda^2 + 5$

$= \lambda^2 + 2\lambda - 3$

The eigenvalues are the solutions to $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. You begin by finding $\mathbf{A} - \lambda\mathbf{I}$, and then finding its determinant as a polynomial in λ.

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow \lambda^2 + 2\lambda - 3 = 0$

This equation is the **characteristic equation** of **A**.

$(\lambda - 1)(\lambda + 3) = 0$

$\lambda = 1$ or -3

The eigenvalues of **A** are 1 and −3.

Find an eigenvector of A corresponding to the eigenvalue 1:

$\begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$

An eigenvector is a solution to $\mathbf{Ax} = \lambda\mathbf{x}$. In this case, you have to find a column vector $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfying the equation when $\lambda = 1$.

$\begin{pmatrix} 2x + 5y \\ -x - 4y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements,

Equating the lower elements gives $-x - 4y = y$, which leads to $x = -5y$. This is the same equation as you obtain from the upper elements and so gives you no extra information. With 2 × 2 matrices, one equation gives sufficient information to find an eigenvector.

$2x + 5y = x$

$\Rightarrow \quad x = -5y$

Let $y = 1$, then $x = -5 \times 1 = -5$

Here you have a free choice of one variable. You can choose any non-zero value of y and then evaluate x. It is sensible to choose a simple number that avoids fractions.

An eigenvector corresponding to 1 is $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$.

Problem-solving

There are infinitely many eigenvectors for any given eigenvalue. Any non-zero scalar multiple of $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$ will also be an eigenvector of **A** with eigenvalue 1.

Find an eigenvector of A corresponding to the eigenvalue −3:

$\begin{pmatrix} 2 & 5 \\ -1 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$

Repeat the procedure used for $\lambda = 1$ with $\lambda = -3$.

$\begin{pmatrix} 2x + 5y \\ -x - 4y \end{pmatrix} = \begin{pmatrix} -3x \\ -3y \end{pmatrix}$

Equating the upper elements,

$2x + 5y = -3x$

$5x + 5y = 0 \Rightarrow y = -x$

Let $x = 1$, then $y = -1$

An eigenvector corresponding to -3 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The lower elements would give $-x - 4y = -3y$, which is equivalent to $y = -x$.

Any multiple of this vector is also an eigenvector of **A** with eigenvalue −3.

You are sometimes asked to find a **normalised** eigenvector.

- **If $\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector of a matrix A, then the unit vector $\hat{\mathbf{a}} = \begin{pmatrix} \frac{a}{|\mathbf{a}|} \\ \frac{b}{|\mathbf{a}|} \end{pmatrix}$ is a normalised eigenvector of A.**

Links For any non-zero vector **a**, the **unit vector** in the direction of **a** is written as $\hat{\mathbf{a}}$.

← Pure Year 1, Chapter 11

In the example above, the normalised eigenvectors are

$$\begin{pmatrix} \frac{-5}{\sqrt{(-5)^2 + 1^2}} \\ \frac{1}{\sqrt{(-5)^2 + 1^2}} \end{pmatrix} = \begin{pmatrix} -\frac{5}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{\sqrt{1^2 + (-1)^2}} \\ \frac{-1}{\sqrt{1^2 + (-1)^2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Sometimes, the characteristic equation has a single repeated solution or no real solutions. This leads to either **repeated eigenvalues** or **complex eigenvalues**.

Example 2

Find the eigenvalues and corresponding eigenvectors for these matrices:

a $\mathbf{A} = \begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ **c** $\mathbf{C} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$

a $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{pmatrix}$

$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -18 \\ 2 & -9-\lambda \end{vmatrix}$

$= (3-\lambda)(-9-\lambda) - (-18)(2)$

$= \lambda^2 + 6\lambda + 9$

$\lambda^2 + 6\lambda + 9 = 0 \Rightarrow (\lambda + 3)^2 = 0$

Solve the characteristic equation for matrix **A**.

Hence $\lambda = -3$ is a repeated eigenvalue.

Find the corresponding eigenvector(s):

$\begin{pmatrix} 3 & -18 \\ 2 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements,

$3x - 18y = -3x \Rightarrow x = 3y$

So a corresponding eigenvector is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Setting $y = 1$ gives $x = 3$.

b $\mathbf{B} - \lambda\mathbf{I} = \begin{pmatrix} -3-\lambda & 0 \\ 0 & -3-\lambda \end{pmatrix}$

$\det(\mathbf{B} - \lambda\mathbf{I}) = \begin{vmatrix} -3-\lambda & 0 \\ 0 & -3-\lambda \end{vmatrix}$

$= (-3-\lambda)(-3-\lambda)$

$= (-3-\lambda)^2$

Hence $\lambda = -3$ is a repeated eigenvalue.

Find the corresponding eigenvector(s):

$\begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements,

$-3x = -3x$

This equation does not establish a relationship between x and y.

Hence x and y can take any arbitrary value and every vector is an eigenvector of **B**.

The simplest pair of linearly independent eigenvectors is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

c $\mathbf{C} - \lambda\mathbf{I} = \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}$

$\det(\mathbf{C} - \lambda\mathbf{I}) = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix}$

$= (3-\lambda)(-1-\lambda) - (-2)(4)$

$= \lambda^2 - 2\lambda + 5$

$\lambda^2 - 2\lambda + 5 = 0$

$(\lambda - 1)^2 + 4 = 0$

$\lambda = 1 \pm 2i$

Find the corresponding eigenvectors:

For $\lambda = 1 + 2i$, $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (1+2i)\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements,

$3x - 2y = (1+2i)x \Rightarrow (2-2i)x = 2y$

Set $x = 1$, so $2 - 2i = 2y \Rightarrow y = 1 - i$

So the eigenvector corresponding to $1 + 2i$ is $\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$.

Set $x = 1$, so $2 + 2i = 2y \Rightarrow y = 1 + i$

So the eigenvector corresponding to $1 - 2i$ is $\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$.

Notation A set of vectors is **linearly independent** if no vector in the set can be written as a linear combination of the others. For a set of two vectors, this means that one cannot be written as a scalar multiple of the other.

Similarly, equating the lower elements leads to $-3y = -3y$.

You could give **any** two linearly independent vectors as your eigenvectors but it is convention to choose $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

This matrix has two distinct complex eigenvalues. Note that if a matrix with real elements has complex eigenvalues, they will occur in a conjugate pair. ← Core Pure Book 1, Chapter 1

Problem-solving

The eigenvectors corresponding to complex eigenvalues can be written with one real and one complex element. In this form, the complex elements will be conjugates of each other.

Example 3

A transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$$

a Find the eigenvalues of **A**.

b Find Cartesian equations of the two lines passing through the origin which are invariant under T.

a $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix}$

$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix}$

$= (4-\lambda)(-2-\lambda) + 5$

$= -8 - 4\lambda + 2\lambda + \lambda^2 + 5$

$= \lambda^2 - 2\lambda - 3$

$= (\lambda - 3)(\lambda + 1)$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow \lambda = 3$ or -1

The eigenvalues of **A** are 3 and −1.

> With practice, you can write down this line without the previous working.

b $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements,

$4x - 5y = 3x \Rightarrow x = 5y$

So an eigenvector is $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

The line through the origin in the direction of $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ is invariant under T.

The equation of this line is $y = \frac{1}{5}x$.

$\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -1\begin{pmatrix} x \\ y \end{pmatrix}$

Equating the upper elements,

$4x - 5y = -x \Rightarrow x = y$

So an eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The line through the origin in the direction of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is invariant under T.

The equation of this line is $y = x$.

> **Problem-solving**
> Find the eigenvectors. The directions of the eigenvectors are not changed by the transformation, so each eigenvector will specify an invariant line through the origin.

> Check your answer:
> $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 5y \\ y \end{pmatrix} = \begin{pmatrix} 20y - 5y \\ 5y - 2y \end{pmatrix} = \begin{pmatrix} 15y \\ 3y \end{pmatrix}$ ✓
> $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} 4x - 5x \\ x - 2x \end{pmatrix} = \begin{pmatrix} -x \\ -x \end{pmatrix}$ ✓

Exercise 5A

1 Find the eigenvalues and corresponding eigenvectors of the matrices

a $\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$ **c** $\begin{pmatrix} 3 & -2 \\ 0 & 4 \end{pmatrix}$

2 Find the eigenvalues and corresponding eigenvectors of the following matrices.

a $\mathbf{M} = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix}$ **b** $\mathbf{N} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Hint For part **b**, give two linearly independent eigenvectors.

3 Find the eigenvalues and corresponding eigenvectors of the following matrices.

a $\mathbf{A} = \begin{pmatrix} -3 & -1 \\ 4 & -3 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$

Hint The eigenvalues will be complex.

(P) 4 A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ -2 & 9 \end{pmatrix}$$

a Find the eigenvalues of **A**.

b Find Cartesian equations of the two lines passing through the origin which are invariant under T.

5 Show that the matrix $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has a repeated eigenvalue and find the corresponding eigenvector.

(E/P) 6 The matrix $\mathbf{A} = \begin{pmatrix} 3 & k \\ 1 & -1 \end{pmatrix}$ has a repeated eigenvalue.

Find the value of k. **(4 marks)**

(E/P) 7 The matrix $\mathbf{M} = \begin{pmatrix} 1 & -1 \\ k & -3 \end{pmatrix}$ has complex eigenvalues.

Find the set of possible values of k. **(4 marks)**

(E/P) 8 Show that any 2 × 2 matrix of the form $\mathbf{A} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ $a, b \in \mathbb{R}$, has eigenvalues $a \pm b\mathrm{i}$. **(3 marks)**

(E/P) 9 The linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ maps the point (5, 2) to the point (−15, −6).
Write down an eigenvector of the matrix representing T and its corresponding eigenvalue.
(3 marks)

(E/P) **10** **a** Show that the matrix $\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ has no real eigenvalues and the corresponding linear transformation has no invariant lines. **(4 marks)**

b Hence explain why the corresponding linear transformation has no invariant lines. **(1 mark)**

(P) **11** The matrix $\mathbf{M} = \begin{pmatrix} -\frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$ represents a linear transformation T.

a Find the eigenvalues and corresponding eigenvectors of this matrix. **(3 marks)**

b Show that the eigenvectors are perpendicular. **(2 marks)**

c Explain why every point on the line $y = 2x$ is invariant. **(1 mark)**

d Fully describe the transformation T. **(3 marks)**

(E/P) **12** Show that that if λ is an eigenvalue of matrix $\mathbf{A}$, then λ^2 is an eigenvalue of $\mathbf{A}^2$. **(3 marks)**

(P) **13** M is a 2×2 matrix, and $\mathbf{x}$ is a non-zero vector.
Given that $\mathbf{Mx} = 0$, show that $\mathbf{M}$ is singular. **(3 marks)**

Challenge

The linear transformation T is represented by the matrix $\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$.

Explain why T has infinitely many invariant lines, and fully describe all such invariant lines.

A You can also find the eigenvalues and eigenvectors of a 3×3 matrix.

3×3 matrices have cubic characteristic equations. Often questions will give you a hint which will help you to factorise the cubic. However, if a hint is not given, you may have to search for one of the eigenvalues using the factor theorem.

Links The **factor theorem** states that, for a polynomial $f(x)$, $f(p) = 0$ if and only if $(x - p)$ is a factor of $f(x)$. ← **Pure Year 1, Chapter 7**

Example 4

Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -4 & -3 \end{pmatrix}$.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -4 & -3 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -4 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 & -3 \\ 0 & 2-\lambda & 1 \\ 0 & -4 & -3-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 & -3 \\ 0 & 2-\lambda & 1 \\ 0 & -4 & -3-\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ -4 & -3-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & -3-\lambda \end{vmatrix} + (-3)\begin{vmatrix} 0 & 2-\lambda \\ 0 & -4 \end{vmatrix}$$

$$= (2-\lambda)((2-\lambda)(-3-\lambda) + 4) - 0 + 0$$

$$= (2-\lambda)(-6 - 2\lambda + 3\lambda + \lambda^2 + 4)$$

$$= (2-\lambda)(\lambda^2 + \lambda - 2)$$

$$= (2-\lambda)(\lambda + 2)(\lambda - 1)$$

$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (2-\lambda)(\lambda+2)(\lambda-1) = 0$

$\Rightarrow \lambda = 2, -2$ or 1

The eigenvalues of **A** are −2, 1 and 2.

As with 2 × 2 matrices, the eigenvalues are the solutions to $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$. You begin by finding $\mathbf{A} - \lambda\mathbf{I}$ and finding its determinant. With a 3 × 3 matrix the characteristic equation is a cubic which will have 3 roots and hence 3 eigenvalues.

Find an eigenvector of A corresponding to the eigenvalue −2:

$$\begin{pmatrix} 2 & 1 & -3 \\ 0 & 2 & 1 \\ 0 & -4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

An eigenvector is a solution to $\mathbf{Ax} = \lambda\mathbf{x}$. In this case, you have to find a column vector $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ satisfying the equation when $\lambda = -2$.

$$\begin{pmatrix} 2x + y - 3z \\ 2y + z \\ -4y - 3z \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

Equating the middle elements,

$2y + z = -2y \Rightarrow z = -4y$

Let $y = 1$, then $z = -4$.

Here you have a free choice of one variable. You can choose any non-zero value for y or x and then evaluate the other variable.

Equating the top elements and substituting $y = 1$ and $z = -4$,

$2x + y - 3z = -2x$

$4x = -y + 3z$

$= -1 - 12 = -13 \Rightarrow x = -\frac{13}{4}$

Equating the lowest elements gives an equivalent equation to the one you obtained from the middle elements and so gives you no extra information. With 3 × 3 matrices, usually two equations will give you all the information you need to find an eigenvector.

An eigenvector corresponding to −2 is $\begin{pmatrix} -\frac{13}{4} \\ 1 \\ -4 \end{pmatrix}$.

Find an eigenvector of **A** corresponding to the eigenvalue 1:

$$\begin{pmatrix}2 & 1 & -3\\0 & 2 & 1\\0 & -4 & -3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 1\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}2x + y - 3z\\2y + z\\-4y - 3z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix}$$

Equating the middle elements,

$$2y + z = y \Rightarrow y = -z$$

Let $z = 1$, then $y = -1$.

Equating the top elements and substituting $y = -1$ and $z = 1$,

$$2x + y - 3z = x$$
$$x = -y + 3z = 1 + 3 = 4$$

An eigenvector corresponding to 1 is $\begin{pmatrix}4\\-1\\1\end{pmatrix}$.

Repeat the procedure with $\lambda = 1$.

Any non-zero multiple of this eigenvector would also be a correct eigenvector.

Find an eigenvector of **A** corresponding to the eigenvalue 2:

$$\begin{pmatrix}2 & 1 & -3\\0 & 2 & 1\\0 & -4 & -3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 2\begin{pmatrix}x\\y\\z\end{pmatrix}$$

$$\begin{pmatrix}2x + y - 3z\\2y + z\\-4y - 3z\end{pmatrix} = \begin{pmatrix}2x\\2y\\2z\end{pmatrix}$$

Equating the middle elements,

$$2y + z = 2y \Rightarrow z = 0$$

Equating the bottom elements and using $z = 0$,

$$-4y - 3z = 2z \Rightarrow 4y = -5z = 0 \Rightarrow y = 0$$

Equating the top elements,

$$2x + y - 3z = 2x \Rightarrow y = 3z \Rightarrow y = 0, z = 0$$

Let $x = 1$

An eigenvector corresponding to 2 is $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

This calculation differs from the calculation for the other two eigenvalues in that these two equations give you that $y = z = 0$ and there is no choice of values.

The variable x appears in no equation and so can take any value. 1 is the simplest value to take.

A 3 × 3 matrix will always have at least one real eigenvalue since a cubic equation always has at least one real solution.

Example 5

The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & 0 \end{pmatrix}$

a Show that -2 is the only real eigenvalue of **A**.

b Find a normalised eigenvector of **A** corresponding to the eigenvalue -2.

a $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & -1 & 1 \\ 0 & 3-\lambda & 5 \\ 2 & 1 & -\lambda \end{pmatrix}$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & -1 & 1 \\ 0 & 3-\lambda & 5 \\ 2 & 1 & -\lambda \end{vmatrix}$$

$$= (2-\lambda)\begin{vmatrix} 3-\lambda & 5 \\ 1 & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} 0 & 5 \\ 2 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 3-\lambda \\ 2 & 1 \end{vmatrix}$$

$$= (2-\lambda)(-3\lambda + \lambda^2 - 5) - 10 - 2(3-\lambda)$$

$$= -6\lambda + 2\lambda^2 - 10 + 3\lambda^2 - \lambda^3 + 5\lambda - 10 - 6 + 2\lambda$$

$$= -\lambda^3 + 5\lambda^2 + \lambda - 26$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow -\lambda^3 + 5\lambda^2 + \lambda - 26 = 0$$

$$\lambda^3 - 5\lambda^2 - \lambda + 26 = 0$$

$$(\lambda + 2)(\lambda^2 + k\lambda + 13) = 0$$

Equating coefficients of x^2,

$$-5 = 2 + k \Rightarrow k = -7$$

$$(\lambda + 2)(\lambda^2 - 7\lambda + 13) = 0$$

The question implies that $\lambda = -2$ is a root of the characteristic equation and so $(\lambda + 2)$ must be a factor of the cubic. Equating a coefficient has been used here to complete the factorisation but you can use any appropriate method.

The discriminant of $\lambda^2 - 7\lambda + 13 = 0$ is

$$b^2 - 4ac = 49 - 52 = -3 < 0$$

To show that there is only one real root of the cubic, show that the discriminant of the quadratic factor is negative.

The quadratic factor $\lambda^2 - 7\lambda + 13$ has no real roots.

So -2 is the only real eigenvalue of **A**.

b $\begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 5 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} 2x - y + z \\ 3y + 5z \\ 2x + y \end{pmatrix} = \begin{pmatrix} -2x \\ -2y \\ -2z \end{pmatrix}$$

Equating the middle elements,

$$3y + 5z = -2y \Rightarrow y = -z$$

Let $z = 1$, then $y = -1$.

Equating the bottom elements,

$$2x + y = -2z \Rightarrow x = \frac{-y - 2z}{2}$$

Substituting $y = -1$ and $z = 1$,

$$x = \frac{1-2}{2} = -\frac{1}{2}$$

An eigenvector of $\mathbf{A}$ is $2\begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$

The working gives $\begin{pmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{pmatrix}$ as the eigenvector but, as any multiple of this is also an eigenvector, it is sensible to multiply this by 2, or −2, to avoid working in fractions.

The magnitude of this eigenvector is $\sqrt{(-1)^2 + (-2)^2 + 2^2} = 3$

A normalised eigenvector is found by dividing all of the terms by the magnitude of the original eigenvector.

A normalised eigenvector of $\mathbf{A}$ is

$$\frac{1}{3}\begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{pmatrix}$ would also be correct. If a column vector $\mathbf{x}$ is a normalised eigenvector of a matrix, then $-\mathbf{x}$ is also a normalised eigenvector.

Example 6

The matrix $\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

a Show that 1 is a repeated eigenvalue of $\mathbf{A}$ and find the other distinct eigenvalue.

b Find two linearly independent eigenvectors corresponding to the eigenvalue 1.

a $\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$

$$= (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 2-\lambda \\ 1 & 1 \end{vmatrix}$$

$$= (2-\lambda)((2-\lambda)^2 - 1) - (1-\lambda) + (\lambda - 1)$$

$$= (2-\lambda)(\lambda^2 - 4\lambda + 3) + 2\lambda - 2$$

$$= (2-\lambda)(\lambda - 3)(\lambda - 1) + 2(\lambda - 1)$$

$$= (\lambda - 1)(2\lambda - 6 - \lambda^2 + 3\lambda + 2)$$

$$= -(\lambda - 1)(\lambda^2 - 5\lambda + 4) = -(\lambda - 1)^2(\lambda - 4)$$

Factorise the cubic. Since you know that 1 is a root of the cubic, look for the factor $(\lambda - 1)$.

The solutions to $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ are 1 repeated and 4.

So 1 is a repeated eigenvalue and 4 is the other eigenvalue.

b $\begin{pmatrix} 2x + y + z \\ x + 2y + z \\ x + y + 2z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Equating the top elements,

$$2x + y + z = x \Rightarrow x + y + z = 0$$

Equating the middle and bottom elements both give you the same equation so the elements of the eigenvectors only need to satisfy this one equation.

A

To find two linearly independent eigenvectors,

$y = 0$ and $z = 1 \Rightarrow x = -1$

$y = 1$ and $z = 0 \Rightarrow x = -1$

So two linearly independent eigenvectors are $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

The choices of $y = 0$ and $z = 1$, and $y = 1$ and $z = 0$ are arbitrary but guarantee that the eigenvectors are linearly independent.

Watch out Not all repeated eigenvectors will give two linearly independent eigenvectors.

Exercise 5B

1 Find the eigenvalues and corresponding eigenvectors of the matrices

a $\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & 2 \\ -2 & 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 4 & -2 & -4 \\ 2 & 3 & 0 \\ 2 & -5 & -4 \end{pmatrix}$

E/P **2** $\mathbf{M} = \begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix}$

a Show that −2 is a repeated eigenvalue of **M** and find the other distinct eigenvalue. **(4 marks)**

b Find two linearly independent eigenvectors corresponding to the eigenvalue −2. **(3 marks)**

3 $\mathbf{A} = \begin{pmatrix} 3 & -1 & 2 \\ 3 & -1 & 6 \\ -2 & 2 & -2 \end{pmatrix}$

Find the eigenvalues and corresponding eigenvectors for matrix **A**.

4 The matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & -2 \\ -3 & 2 & 0 \\ 1 & 4 & -3 \end{pmatrix}$

a Show that −1 is the only real eigenvalue of **A**.

b Find an eigenvector corresponding to the eigenvalue −1.

c Find the two complex eigenvalues and their corresponding eigenvectors.

Hint Find the roots of the quadratic factor in the characteristic equation for matrix **A**.

E **5** The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 2 & 4 \\ 0 & 2 & 0 \end{pmatrix}$

a Show that 4 is an eigenvalue of **A** and find the other two eigenvalues of **A**. **(4 marks)**

b Find an eigenvector corresponding to the eigenvalue 4. **(2 marks)**

A E **6** The matrix $\mathbf{A} = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -1 \\ 4 & 4 & 3 \end{pmatrix}$

Given that 3 is an eigenvalue of **A**,

a find the other two eigenvalues of **A** **(4 marks)**

b find the eigenvector corresponding to each of the eigenvalues of **A**. **(4 marks)**

E **7** The matrix $\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ -2 & 4 & 0 \\ 4 & 2 & 5 \end{pmatrix}$

a Show that 2 is an eigenvalue of **A**. **(2 marks)**

b Find the other two eigenvalues of **A**. **(2 marks)**

c Find a normalised eigenvector of **A** corresponding to the eigenvalue 2. **(2 marks)**

E **8** The matrix $\mathbf{A} = \begin{pmatrix} 4 & 2 & 1 \\ -2 & 0 & 5 \\ 0 & 3 & 4 \end{pmatrix}$

a Show that −2 is an eigenvalue of **A** and that there is only one other eigenvalue. **(4 marks)**

b Find an eigenvector corresponding to each of the eigenvalues. **(4 marks)**

E **9** The matrix $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 1 & 2 & 1 \end{pmatrix}$.

Given that 2 is an eigenvalue of **A**,

a find the other two eigenvalues of **A** **(4 marks)**

b find the eigenvector corresponding to each of the eigenvalues of **A**. **(4 marks)**

E/P **10** Given that $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ is an eigenvector of the matrix **A** where $\mathbf{A} = \begin{pmatrix} 4 & 1 & 2 \\ 1 & a & 0 \\ -1 & 1 & b \end{pmatrix}$

a find the eigenvalue of **A** corresponding to $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ **(2 marks)**

b find the value of a and the value of b **(4 marks)**

c show that **A** has only one real eigenvalue. **(2 marks)**

d Find the two complex eigenvalues and their corresponding eigenvectors. **(6 marks)**

E/P **11** $\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 4 & -1 & 3 \end{pmatrix}$

a Find the eigenvalues of matrix **A** and hence find a set of eigenvectors. **(6 marks)**

Matrix **A** represents the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$.

b Find vector equations of the invariant lines under T. **(4 marks)**

E/P **12** Explain why every linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ must have at least one invariant line. **(2 marks)**

Challenge

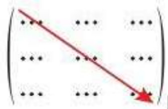
A The matrix $\mathbf{M} = \begin{pmatrix} \frac{1}{9} & \frac{8}{9} & -\frac{4}{9} \\ \frac{8}{9} & \frac{1}{9} & \frac{4}{9} \\ -\frac{4}{9} & \frac{4}{9} & \frac{7}{9} \end{pmatrix}$ represents a reflection in plane Π.

Find the eigenvalues and eigenvectors of **M** and hence find the Cartesian equation of the plane Π.

5.2 Reducing matrices to diagonal form

Calculations with matrices can often be simplified by reducing a matrix to a given form. In this section you will learn how to reduce some matrices to **diagonal** form.

- **A diagonal matrix is a square matrix in which all of the elements which are not on the diagonal from the top left to the bottom right of the matrix are zero. The diagonal from the top left to the bottom right of the matrix is called the leading diagonal.**



leading diagonal

- **The general 2 × 2 diagonal matrix is $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.**

For example, $\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$, $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and (1) are all diagonal matrices.

Watch out Any non-zero elements must be on the **leading diagonal**. For example, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ are not diagonal matrices.

- **To reduce a given matrix A to diagonal form, use the following procedure.**
  - **Find the eigenvalues and eigenvectors of A.**
  - **Form a matrix P which consists of the eigenvectors of A.**
  - **Find $\mathbf{P}^{-1}$.**
  - **A diagonal matrix D is given by $\mathbf{P}^{-1}\mathbf{AP}$.**

Note A matrix which can be reduced to diagonal form in this way is called a **diagonalisable** matrix. Not every matrix can be diagonalised, although any $n \times n$ matrix with n distinct eigenvalues can be. In your exam you will only be asked to diagonalise matrices of this type.

Example 7

$\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$

a Find a matrix **P** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$ is diagonal.

b Write down the diagonal matrix **D**.

a $\begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 \times 1$

$= \lambda^2 - 7\lambda + 10$

$\lambda^2 - 7\lambda + 10 = 0 \Rightarrow \lambda = 5 \text{ or } 2$

Find the eigenvalues of **A**.

$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

For $\lambda = 2$, $4x + 2y = 2x$ so $x = -y$

A corresponding eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

For $\lambda = 5$, $4x + 2y = 5x$ so $2y = x$

A corresponding eigenvector $= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Find the eigenvectors of **A**.

So $\mathbf{P} = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$

b $\det \mathbf{P} = -1 \times 1 - 2 \times 1 = -3$

Hence $\mathbf{P}^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}$

Find $\mathbf{P}^{-1}$. Remember that for any non-singular matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

← Core Pure Book 1, Chapter 6

$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$

$\mathbf{D} = -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$

$= -\frac{1}{3}\begin{pmatrix} 1 & -2 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} -2 & 10 \\ 2 & 5 \end{pmatrix}$

$= -\frac{1}{3}\begin{pmatrix} -6 & 0 \\ 0 & -15 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix}$

The matrix **D** consists of the eigenvalues of **A** along the leading diagonal in the same order as the eigenvectors are given in matrix **P**. All other elements are zero as required.

- **When you reduce a matrix A to a diagonal matrix D, the elements on the diagonal are the eigenvalues of A.**

The above process for diagonalising a matrix relies on finding the inverse of **P**. For larger matrices this can be a time-consuming process. If a matrix is **symmetric** you can diagonalise it more easily.

- **A matrix, A, is symmetric if $\mathbf{A} = \mathbf{A}^T$.**
The elements of a symmetric matrix are symmetric with respect to the leading diagonal.

Watch out Don't confuse the matrix **P** which diagonalises a given matrix, **A**, with the diagonal matrix **D**. **P** is formed from the **eigenvectors** of **A**, and **D** has the **eigenvalues** of **A** on its leading diagonal.

Links $\mathbf{A}^T$ is the **transpose** of matrix **A**. It is the matrix formed by interchanging the rows and columns of matrix **A**.

← Core Pure Book 1, Section 6.5

For example, $\begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}$ and $\begin{pmatrix} 1 & b & 3 \\ b & a & 0 \\ 3 & 0 & 0 \end{pmatrix}$ are symmetric matrices.

If the matrix **A** is symmetric, you can carry out **orthogonal diagonalisation**.

- **The procedure for orthogonal diagonalisation of a symmetric matrix A is:**
  - **Find the normalised eigenvectors of A.**
  - **Form a matrix P which consists of the normalised eigenvectors of A.**
  - **Write down $\mathbf{P}^{\mathrm{T}}$.**
  - **A diagonal matrix D is given by $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$.**

Note For any symmetric matrix, the normalised eigenvectors are mutually perpendicular. This means that the matrix formed from the normalised eigenvectors has the property that $\mathbf{P}^{-1} = \mathbf{P}^{\mathrm{T}}$. Matrices which have this property are called **orthogonal matrices**.

Example 8

The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Reduce **A** to a diagonal matrix.

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1$$

$$= \lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0 \Rightarrow (\lambda - 1)(\lambda - 3) = 0$$

$$\Rightarrow \lambda = 1 \text{ or } 3$$

To diagonalise a symmetric matrix, you need to find normalised eigenvectors of the matrix, so start by finding the eigenvalues.

$\lambda = 1$:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating the upper elements,

$$2x - y = x \Rightarrow y = x$$

Let $x = 1$, then $y = 1$.

An eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

Watch out If you are using orthogonal diagonalisation, you need to find the **normalised** eigenvectors.

To convert an eigenvector **x** to a normalised eigenvector, divide each of the elements of **x** by the magnitude of **x**.

$\lambda = 3$:

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

Equating the upper elements,

$2x - y = 3x \Rightarrow y = -x$

Let $x = 1$, then $y = -1$.

An eigenvector corresponding to the eigenvalue 3 is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is $\sqrt{1^2 + (-1)^2} = \sqrt{2}$.

A normalised eigenvector corresponding to the eigenvalue 1 is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

The negative of this vector, $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, is also correct and would just as appropriate for diagonalising the matrix.

$$\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Form the orthogonal matrix **P** from the normalised eigenvectors by using the eigenvectors as the columns of the matrix.

$$\mathbf{P}^{\mathrm{T}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

In this case, as **P** is symmetric, $\mathbf{P}^{\mathrm{T}} = \mathbf{P}$.

$$\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{2}{\sqrt{2}} - \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} + \frac{1}{2} & \frac{3}{2} - \frac{3}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{3}{2} + \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$$

The diagonal matrix is given by $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

The non-zero number in the first column, 1, is the eigenvalue corresponding to the eigenvector $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ used as the first column of **P**.

The non-zero number in the second column, 3, is the eigenvalue corresponding to the eigenvector $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ used as the first column of **P**.

If you had taken **P** as $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$, then **D** would be $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

A similar result is true for 3 × 3 matrices.

A The process of diagonalising matrices is the same for 3 × 3 matrices.

Example 9

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 4 & 2 & -3 \\ 4 & 2 & 3 \end{pmatrix}$$

Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$.

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 4 & 2-\lambda & -3 \\ 4 & 2 & 3-\lambda \end{vmatrix} = (1-\lambda)((2-\lambda)(3-\lambda)+6) - (4(3-\lambda)+12) + 2(8-4(2-\lambda))$$
$$= -(\lambda+1)(\lambda-3)(\lambda-4)$$

So eigenvalues are −1, 3 and 4.

$$\begin{pmatrix} 1 & 1 & 2 \\ 4 & 2 & -3 \\ 4 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\lambda = -1$:

Equating the top elements,

$x + y + 2z = -x \Rightarrow 2x + y + 2z = 0$ (1)

Equating the middle elements,

$4x + 2y - 3z = -y \Rightarrow 4x + 3y - 3z = 0$ (2)

Equating the bottom elements,

$4x + 2y + 3z = -z \Rightarrow 4x + 2y + 4z = 0$ (3)

(2) − (3) gives $y - 7z = 0$

Setting $z = 1$ gives $y = 7$ and $x = -\frac{9}{2}$

Note that equation (1) is just a multiple of equation (3), so the system can be solved using just equations (2) and (3).

A corresponding eigenvector is $\begin{pmatrix} -9 \\ 14 \\ 2 \end{pmatrix}$.

It is conventional to give an eigenvector in integer form, so multiply the x, y and z values through by 2 before stating the eigenvector.

$\lambda = 3$:

Equating the top elements,

$x + y + 2z = 3x \Rightarrow -2x + y + 2z = 0$ (1)

Equating the middle elements,

$4x + 2y - 3z = 3y \Rightarrow 4x - y - 3z = 0$ (2)

Equating the bottom elements,

$4x + 2y + 3z = 3z \Rightarrow 4x + 2y = 0$ (3)

Equation (3) gives $y = -2x$

Setting $x = 1$ gives $y = -2$ and $z = 2$.

A corresponding eigenvector is $\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$.

A

$\lambda = 4$:

Equating the top elements,

$x + y + 2z = 4x \Rightarrow -3x + y + 2z = 0$ (1)

Equating the middle elements,

$4x + 2y - 3z = 4y \Rightarrow 4x - 2y - 3z = 0$ (2)

Equating the bottom elements,

$4x + 2y + 3z = 4z \Rightarrow 4x + 2y - z = 0$ (3)

(3) − (2) gives $4y + 2z = 0$

Setting $y = 1$ gives $z = -2$ and $x = -1$.

A corresponding eigenvector is $\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$.

So $\mathbf{P} = \begin{pmatrix} -9 & 1 & -1 \\ 14 & -2 & 1 \\ 2 & 2 & -2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

You can write down matrix **D** by entering the eigenvalues corresponding to the order of the eigenvectors in **P** in the leading diagonal of **D**.

You can check your solution by finding the inverse of **P** using your calculator, and then finding the matrix product $\mathbf{P}^{-1}\mathbf{AP}$.

Example 10

The matrix $\mathbf{A} = \begin{pmatrix} 7 & 5 & 5 \\ 5 & -2 & 4 \\ 5 & 4 & -2 \end{pmatrix}$

a Verify that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of **A** and find the corresponding eigenvalue.

b Show that −6 is another eigenvalue of **A** and find the corresponding eigenvector.

c Given that the third eigenvector of **A** is $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$.

a $\begin{pmatrix} 7 & 5 & 5 \\ 5 & -2 & 4 \\ 5 & 4 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 + 5 + 5 \\ 10 - 2 + 4 \\ 10 + 4 - 2 \end{pmatrix} = \begin{pmatrix} 24 \\ 12 \\ 12 \end{pmatrix} = 12\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Hence $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of **A** corresponding to the eigenvalue 12.

To show that **x** is an eigenvector of **A**, you have to find a constant, λ, such that $\mathbf{Ax} = \lambda\mathbf{x}$.

b $\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 7 & 5 & 5 \\ 5 & -2 & 4 \\ 5 & 4 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

$= \begin{pmatrix} 7 - \lambda & 5 & 5 \\ 5 & -2 - \lambda & 4 \\ 5 & 4 & -2 - \lambda \end{pmatrix}$

When $\lambda = -6$,

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 7-(-6) & 5 & 5 \\ 5 & -2-(-6) & 4 \\ 5 & 4 & -2-(-6) \end{vmatrix} = \begin{vmatrix} 13 & 5 & 5 \\ 5 & 4 & 4 \\ 5 & 4 & 4 \end{vmatrix}$$

$$= 13\begin{vmatrix} 4 & 4 \\ 4 & 4 \end{vmatrix} - 5\begin{vmatrix} 5 & 5 \\ 4 & 4 \end{vmatrix} + 5\begin{vmatrix} 5 & 5 \\ 4 & 4 \end{vmatrix}$$

$$= 13(16 - 16) - 5(20 - 20) + 5(20 - 20) = 0$$

So -6 is an eigenvalue of A.

Find an eigenvector corresponding to -6:

$$\begin{pmatrix} 7 & 5 & 5 \\ 5 & -2 & 4 \\ 5 & 4 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -6\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} 7x + 5y + 5z \\ 5x - 2y + 4z \\ 5x + 4y - 2z \end{pmatrix} = \begin{pmatrix} -6x \\ -6y \\ -6z \end{pmatrix}$$

Equating the top elements,

$$7x + 5y + 5z = -6x \Rightarrow 5y + 5z = -13x$$

$$y + z = -\tfrac{13}{5}x \qquad (1)$$

Equating the middle elements,

$$5x - 2y + 4z = -6y \Rightarrow 4y + 4z = -5x$$

$$y + z = -\tfrac{5}{4}x \qquad (2)$$

From (1) and (2),

$$-\tfrac{13}{5}x = -\tfrac{5}{4}x \Rightarrow x = 0$$

Substituting $x = 0$ into (1),

$$y + z = 0 \Rightarrow y = -z$$

Let $y = 1$, then $z = -1$.

An eigenvector corresponding to the eigenvalue -6 is $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$.

To show that -6 is an eigenvalue, it is sufficient to show that substituting $\lambda = -6$ into $\det(\mathbf{A} - \lambda\mathbf{I})$ gives 0. You do not have to solve the cubic characteristic equation completely.

c Find the eigenvalue corresponding to $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$:

$$\begin{pmatrix} 7 & 5 & 5 \\ 5 & -2 & 4 \\ 5 & 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7-5-5 \\ 5+2-4 \\ 5-4+2 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 3 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

The corresponding eigenvalue is -3.

The magnitude of $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is $\sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$

Watch out The matrix product in part **c** is given as $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$. This tells you that you need to use **orthogonal diagonalisation**. This is possible because the matrix **A** is symmetric.

You will need the eigenvalue corresponding to this eigenvector for the third non-zero element of the diagonal matrix **D**. You already know that the other two elements are 12 and -6.

A

A normalised eigenvector corresponding to 12 is $\begin{pmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ is $\sqrt{0^2 + 1^2 + (-1)^2} = \sqrt{2}$.

A normalised eigenvector corresponding to −6 is $\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$.

The magnitude of $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ is $\sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}$.

A normalised eigenvector corresponding to −3 is $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{pmatrix}$.

So $\mathbf{P} = \begin{pmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 12 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

The matrix **P** is made up of columns of normalised eigenvalues. **P** is an orthogonal matrix and so $\mathbf{P}^{\mathrm{T}} = \mathbf{P}^{-1}$. Hence there is no difference between the expression $\mathbf{P}^{-1}\mathbf{AP}$, used to diagonalise **A** in this example and the expression $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$, used in Example 8.

You know that **P** is a matrix with the normalised eigenvectors as its columns and that **D** is the diagonal matrix with the corresponding eigenvalues as the elements of the leading diagonal. Multiplying the matrices out is a laborious process and you should not do this unless the question requires it.

There are many applications in which diagonal matrices are easier to work with. For example, you can use matrix diagonalisation to solve problems involving coupled differential equations or coupled recurrence relations.

Example 11

The two sequences x_n and y_n satisfy the recurrence relations

$$x_{n+1} = 4x_n + y_n,\ x_1 = 1$$
$$y_{n+1} = 2x_n + 3y_n,\ y_1 = 2 \qquad n \geqslant 1$$

These recurrence relations can be written in matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_n \\ y_n \end{pmatrix},\ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad n \geqslant 1$$

where $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$.

a Find the eigenvalues and corresponding eigenvectors of **A**.

b Hence write down matrices **P** and **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ where **D** is a diagonal matrix.

New sequences u_n and v_n can be formed from x_n and y_n using the transformation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

c Show that $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{D}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$.

d Hence find closed form expressions for the original sequences x_n and y_n.

a $\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2 \times 1$

$= \lambda^2 - 7\lambda + 10$

$= (\lambda - 5)(\lambda - 2)$

So eigenvalues are 5 and 2.

$\begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$

$\lambda = 5$:

Equating the top elements,

$4x + y = 5x \Rightarrow y = x$

A corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\lambda = 2$:

Equating the top elements,

$4x + y = 2x \Rightarrow y = -2x$

A corresponding eigenvector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

b $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$

c $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$

$= \mathbf{P}^{-1}\mathbf{A}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$= \mathbf{P}^{-1}\mathbf{APP}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$

$= \mathbf{DP}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{D}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$ as required.

d $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$

So $u_{n+1} = 5u_n$ and $v_{n+1} = 2v_n$

Hence $u_{n+1} = u_1 \times 5^{n-1}$ and $v_{n+1} = v_1 \times 2^{n-1}$

$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \Rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \mathbf{P}\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$

$x_n = u_n + v_n = u_1 \times 5^{n-1} + v_1 \times 2^{n-1}$

$y_n = u_n - 2v_n = u_1 \times 5^{n-1} - 2v_1 \times 2^{n-1}$

Links **Closed form** means that you need to find x_n and y_n in terms of n only. ← Chapter 4

Use the transformation with n replaced by $n + 1$.

Replace $\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}$ with $\mathbf{A}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$.

Use the fact that $\mathbf{PP}^{-1} = \mathbf{I}$.

Replace $\mathbf{P}^{-1}\mathbf{AP}$ with $\mathbf{D}$.

Use the result from part **c** and your answer for **D** to set up recurrence relations for u and v.

Use standard results for recurrence relations to write down closed forms for u_n and v_n. ← Section 4.2

Use the transformation to set up a relationship between $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ and $\begin{pmatrix} u_n \\ v_n \end{pmatrix}$ in terms of **P**.

Form two equations for x_n and y_n in terms of u_1, v_1 and n.

When $n = 1$,

$1 = u_1 + v_1$

$2 = u_1 - 2v_1$

So $u_1 = \frac{4}{3}$ and $v_1 = -\frac{1}{3}$

Hence:

$x_n = \frac{4}{3}(5^{n-1}) - \frac{1}{3}(2^{n-1})$

$y_n = \frac{4}{3}(5^{n-1}) + \frac{2}{3}(2^{n-1})$

Watch out You know the initial values of x_n and y_n, but you need to find the initial values of u_n and v_n.

Substitute $n = 1$ and solve the resulting simultaneous equations to find u_1 and v_1.

Write the expressions for x_n and y_n in closed form.

Diagonalisation can also be used to compute large powers of matrices efficiently.

- **For a diagonal matrix $\mathbf{D} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $\mathbf{D}^k = \begin{pmatrix} a^k & 0 \\ 0 & d^k \end{pmatrix}$.**

This result holds for diagonal matrices of higher dimension as well.

Consider the matrix product $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ where $\mathbf{D}$ is a diagonal matrix.

Then $\mathbf{A} = \mathbf{PDP}^{-1}$ and $\mathbf{A}^k = (\mathbf{PDP}^{-1})^k$.

$$\begin{aligned}\mathbf{A}^k &= (\mathbf{PDP}^{-1})(\mathbf{PDP}^{-1}) \ldots (\mathbf{PDP}^{-1}) \\ &= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{D}(\mathbf{P}^{-1}\mathbf{P}) \ldots (\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} \\ &= \mathbf{PD}^k\mathbf{P}^{-1}\end{aligned}$$

Hence to calculate $\mathbf{A}^k$, you only need to find matrices $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{P}^{-1}$ and apply the result from above.

Example 12

$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$

a Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

b Hence find $\mathbf{A}^6$.

a Find the eigenvalues and the eigenvectors of $\mathbf{A}$:

$\begin{vmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1 \text{ or } 4$

For $\lambda = 1$, a corresponding eigenvector is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

For $\lambda = 4$, a corresponding eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$\mathbf{P} = \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

$\mathbf{A}^6 = \mathbf{PD}^6\mathbf{P}^{-1}$

$\mathbf{P}^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$

$\mathbf{D}^6 = \begin{pmatrix} 1 & 0 \\ 0 & 4096 \end{pmatrix}$

Find the eigenvalues and the eigenvectors of $\mathbf{A}$ in the usual way.

Use your calculator to find the inverse of matrix $\mathbf{P}$.

Find $\mathbf{D}^6$, $4^6 = 4096$

b $\mathbf{A}^{46} = \frac{1}{3}\begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 4096\end{pmatrix}\begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 8192 & 4096\end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix}8193 & 4095\\ 8190 & 4098\end{pmatrix} = \begin{pmatrix}2731 & 1365\\ 2730 & 1366\end{pmatrix}.$

Compute the matrix product. If you were working with a larger power you could leave your elements in index form.

Exercise 5C

1 For each of these matrices, find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$.

a $\mathbf{A} = \begin{pmatrix}5 & 4\\ 3 & 6\end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix}-3 & -2\\ 5 & 4\end{pmatrix}$

2 Reduce the following matrices to diagonal matrices.

a $\begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}$ **b** $\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}$

(E) **3** $\mathbf{Q} = \begin{pmatrix}4 & -2\\ -1 & 3\end{pmatrix}$

Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$. **(7 marks)**

(E) **4** The matrix $\mathbf{A} = \begin{pmatrix}3 & \sqrt{2}\\ \sqrt{2} & 4\end{pmatrix}$.

a Find the eigenvalues of **A**. **(3 marks)**

b Find normalised eigenvectors of **A** corresponding to each of the two eigenvalues of **A**. **(4 marks)**

c Write down a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$. **(2 marks)**

(E/P) **5** $\mathbf{A} = \begin{pmatrix}7 & 4\\ 4 & 1\end{pmatrix}$

a Show that $\begin{pmatrix}2\\ 1\end{pmatrix}$ and $\begin{pmatrix}-1\\ 2\end{pmatrix}$ are eigenvectors of **A**. **(4 marks)**

Adam says that because **A** is symmetric, the matrix $\mathbf{P} = \begin{pmatrix}2 & -1\\ 1 & 2\end{pmatrix}$ is such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix.

b Explain Adam's mistake, and find a matrix **Q** such that $\mathbf{Q}^{\mathrm{T}}\mathbf{AQ}$ is diagonal. **(3 marks)**

(E) **6** The two sequences x_n and y_n satisfy the recurrence relations

$$x_{n+1} = 2x_n + 4y_n,\ x_1 = 3$$
$$y_{n+1} = 3x_n + y_n,\ y_1 = 1 \qquad n \geqslant 1$$

These recurrence relations can be written in matrix form as

$$\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix} = \mathbf{A}\begin{pmatrix}x_n\\ y_n\end{pmatrix},\ \begin{pmatrix}x_1\\ y_1\end{pmatrix} = \begin{pmatrix}3\\ 1\end{pmatrix} \qquad n \geqslant 1$$

where $\mathbf{A} = \begin{pmatrix}2 & 4\\ 3 & 1\end{pmatrix}$.

a Find the eigenvalues and corresponding eigenvectors of **A**. **(5 marks)**

b Hence write down matrices **P** and **D** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$ where **D** is a diagonal matrix. **(2 marks)**

New sequences u_n and v_n can be formed from x_n and y_n using the transformation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

c Show that $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{D}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$. **(2 marks)**

d Hence find closed form expressions for the original sequences x_n and y_n. **(5 marks)**

E/P **7** $\mathbf{M} = \begin{pmatrix} 3 & -1 \\ 2 & 0 \end{pmatrix}$

a Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P} = \mathbf{D}$. **(7 marks)**

b Hence, or otherwise, find $\mathbf{M}^{100}$, giving each element in terms of suitable powers of 2. **(5 marks)**

A **8** For each of these matrices, find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$.

a $\mathbf{A} = \begin{pmatrix} 1 & 4 & -1 \\ -1 & 6 & -1 \\ 2 & -2 & 4 \end{pmatrix}$ **b** $\mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$

Hint You can use your calculator to find $\mathbf{P}^{-1}$.

E **9** The matrix **M** is given by

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

a Explain why matrix **M** is not orthogonally diagonalisable. **(1 mark)**

b Show that 3 is an eigenvalue of **M** and find the other two eigenvalues. **(4 marks)**

c For each of the eigenvalues, find a corresponding eigenvector. **(4 marks)**

d Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{M}\mathbf{P}$ is a diagonal matrix. **(2 marks)**

P **10** The matrix $\mathbf{P} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$.

a Show that **P** is an orthogonal matrix.

The matrix $\mathbf{A} = \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} & 1 \\ -\frac{3}{2} & \frac{3}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

b Show that $\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{P}$ is a diagonal matrix.

A P **11** The matrix $\mathbf{A} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix}$. Reduce **A** to a diagonal matrix.

E **12** The matrix $\mathbf{A} = \begin{pmatrix} 5 & 3 & 3 \\ 3 & 1 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

The eigenvalues of **A** are 0, −1 and 8.

a Find a normalised eigenvector corresponding to the eigenvalue 0. **(2 marks)**

Given that $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of **A** corresponding to the eigenvalue −1 and that $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector of **A** corresponding to the eigenvalue 8,

b find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. **(3 marks)**

E **13** The matrix $\mathbf{A} = \begin{pmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{pmatrix}$.

a Given that 9 is an eigenvalue of **A**, find the other two eigenvalues of **A**. **(4 marks)**

b Find eigenvectors of **A** corresponding to each of the three eigenvalues of **A**. **(4 marks)**

c Show that the eigenvectors found in part **b** are mutually perpendicular. **(2 marks)**

d Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \mathbf{D}$. **(2 marks)**

E **14** The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{pmatrix}$

a Show that 4 is an eigenvalue of **A** and find the other two eigenvalues of **A**. **(4 marks)**

b Find a normalised eigenvector of **A** corresponding to the eigenvalue 4. **(3 marks)**

Given that $\begin{pmatrix} -2 \\ 3 \\ -\sqrt{5} \end{pmatrix}$ and $\begin{pmatrix} \sqrt{5} \\ 0 \\ -2 \end{pmatrix}$ are eigenvectors of **A**,

c find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. **(3 marks)**

E **15** The eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & -3 \\ 2 & 2 & 3 \\ -3 & 3 & 3 \end{pmatrix}$$

are $\lambda_1, \lambda_2, \lambda_3$, where $\lambda_1 > \lambda_2 > \lambda_3$.

a Show that $\lambda_1 = 6$ and find the other two eigenvalues λ_2 and λ_3. **(4 marks)**

b Verify that $|\mathrm{A}| = \lambda_1\lambda_2\lambda_3$. **(2 marks)**

c Find an eigenvector corresponding to the eigenvalue $\lambda_1 = 6$. **(2 marks)**

A

Given that $\begin{pmatrix}1\\1\\0\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ are eigenvectors corresponding to λ_2 and λ_3,

d write down a matrix **P** such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix. **(3 marks)**

Challenge

A closed ecosystem has a population of kingfishers and a population of fish. The number of kingfishers, x, and the number of fish, y, at time t years are modelled using the differential equations

$$x' = -0.3x - 0.2y$$
$$y' = -0.1x + 0.4y \qquad (1)$$

At time $t = 0$, the number of kingfishers is 5 and the number of fish is 20.

This information can be written in matrix form as

$$\begin{pmatrix}x'\\y'\end{pmatrix} = \mathbf{A}\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x_0\\y_0\end{pmatrix} = \begin{pmatrix}5\\20\end{pmatrix}$$

where $\mathbf{A} = \begin{pmatrix}0.3 & -0.2\\-0.1 & 0.4\end{pmatrix}$.

a Find the eigenvalues and corresponding eigenvectors of **A**.

b Hence write down matrices **P** and **D** such that $\mathbf{PDP}^{-1} = \mathbf{A}$, where **D** is a diagonal matrix.

New variables u and v can be formed from x and y using the transformation

$$\begin{pmatrix}u\\v\end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix}x\\y\end{pmatrix}$$

c Show that $\begin{pmatrix}u'\\v'\end{pmatrix} = \mathbf{D}\begin{pmatrix}u\\v\end{pmatrix}$.

d Show that $u = c_1\mathrm{e}^{0.5t}$ and that $v = c_2\mathrm{e}^{0.2t}$ where c_1 and c_2 are unknown constants.

d Hence solve the system (1) of differential equations.

5.3 The Cayley–Hamilton theorem

Consider the matrix $\mathbf{M} = \begin{pmatrix}1 & 3\\4 & 2\end{pmatrix}$.

The characteristic equation for this matrix is

$$\begin{vmatrix}1-\lambda & 3\\4 & 2-\lambda\end{vmatrix} = 0 \Rightarrow (1-\lambda)(2-\lambda) - 12 = 0 \Rightarrow -10 - 3\lambda + \lambda^2 = 0$$

and $\mathbf{M}^2 = \begin{pmatrix}1 & 3\\4 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\4 & 2\end{pmatrix} = \begin{pmatrix}13 & 9\\12 & 16\end{pmatrix}$

Now consider the matrix expression

$$-10\mathbf{I} - 3\mathbf{M} + \mathbf{M}^2$$

where **I** is the identity matrix and **M** is the matrix above.

$$-10\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - 3\begin{pmatrix}1 & 3\\4 & 2\end{pmatrix} + \begin{pmatrix}13 & 9\\12 & 16\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$$

Hence $-10\mathbf{I} - 3\mathbf{M} + \mathbf{M}^2 = \mathbf{0}$, the zero matrix.

This result illustrates the **Cayley–Hamilton theorem**.

- **The Cayley–Hamilton theorem states that every square matrix M satisfies its own characteristic equation.**

Example 13

Demonstrate that the matrix $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 3 & 3 \end{pmatrix}$ satisfies its own characteristic equation.

$\begin{vmatrix} 5-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = (5-\lambda)(3-\lambda) - 6 = 9 - 8\lambda + \lambda^2$ — Find the characteristic equation of **A**.

$\mathbf{A}^2 = \begin{pmatrix} 5 & 2 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 31 & 16 \\ 24 & 15 \end{pmatrix}$ — Find $\mathbf{A}^2$.

$9\mathbf{I} - 8\mathbf{A} + \mathbf{A}^2 = 9\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 8\begin{pmatrix} 5 & 2 \\ 3 & 3 \end{pmatrix} + \begin{pmatrix} 31 & 16 \\ 24 & 15 \end{pmatrix}$

$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ as required. — Substitute **I**, **A** and $\mathbf{A}^2$ into the characteristic equation and show that it equals **0**.

Example 14

Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix}$$

a find the characteristic equation of **A**.

b Hence show that $\mathbf{A}^3 = 13\mathbf{A} - 18\mathbf{I}$.

a $\begin{vmatrix} 1-\lambda & 2 \\ 3 & -3-\lambda \end{vmatrix} = (1-\lambda)(-3-\lambda) - 6$

So characteristic equation is $\lambda^2 + 2\lambda - 9 = 0$.

b By the Cayley–Hamilton theorem,

$\mathbf{A}^2 = 9\mathbf{I} - 2\mathbf{A}$ (1) — Use the Cayley–Hamilton theorem to produce an equation in **A** and **I**.

$\mathbf{A}^3 = 9\mathbf{A} - 2\mathbf{A}^2$ (2) — Multiply both sides of (1) by **A**.

Substitute (1) into (2):

$\mathbf{A}^3 = 9\mathbf{A} - 2(9\mathbf{I} - 2\mathbf{A})$ — Substitute for $\mathbf{A}^2$.

Hence $\mathbf{A}^3 = 13\mathbf{A} - 18\mathbf{I}$ as required.

Example 15

Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}$$

a find the characteristic equation of **M**.

b Hence, by using the Cayley–Hamilton theorem, find $\mathbf{M}^{-1}$.

a $\begin{vmatrix} 1-\lambda & 2 & 1 \\ -1 & -\lambda & 2 \\ 2 & 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 2) - 2(\lambda - 4) + (-1 + 2\lambda)$

$= -\lambda^3 + \lambda^2 + 2\lambda + 5$

So the characteristic equation is $\lambda^3 - \lambda^2 - 2\lambda - 5 = 0$.

b By the Cayley–Hamilton theorem,

$\mathbf{M}^3 - \mathbf{M}^2 - 2\mathbf{M} = 5\mathbf{I}$

Use the Cayley–Hamilton theorem.

$\Rightarrow \mathbf{M}^2 - \mathbf{M} - 2\mathbf{I} = 5\mathbf{M}^{-1}$ (1)

Multiply by $\mathbf{M}^{-1}$. Remember that $\mathbf{MM}^{-1} = \mathbf{I}$ and $\mathbf{IM}^{-1} = \mathbf{M}^{-1}$.

$$\mathbf{M}^2 = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 0 & -1 \\ 1 & 4 & 4 \end{pmatrix}$$

$$\mathbf{M}^2 - \mathbf{M} - 2\mathbf{I} = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 0 & -1 \\ 1 & 4 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 & 4 \\ 4 & -2 & -3 \\ -1 & 3 & 2 \end{pmatrix}$$

Hence $\mathbf{M}^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 1 & 4 \\ 4 & -2 & -3 \\ -1 & 3 & 2 \end{pmatrix}$

Find $\mathbf{M}^2$, substitute into equation (1) and solve for $\mathbf{M}^{-1}$.

Exercise 5D

1 Demonstrate that the following matrices satisfy their own characteristic equations:

a $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} 7 & -4 \\ 0 & 3 \end{pmatrix}$

(E) **2** Given that

$$\mathbf{A} = \begin{pmatrix} 6 & 2 \\ -1 & 3 \end{pmatrix}$$

a find the characteristic equation of **A**. **(2 marks)**

b Hence show that $\mathbf{A}^3 = 61\mathbf{A} - 180\mathbf{I}$. **(3 marks)**

E **3** Given that

$$\mathbf{M} = \begin{pmatrix} 4 & -2 \\ 0 & 6 \end{pmatrix}$$

a find the characteristic equation of **M**. **(2 marks)**

b Hence, use the Cayley–Hamilton theorem to find $\mathbf{M}^{-1}$. **(3 marks)**

E/P **4** $\mathbf{A} = \begin{pmatrix} 6 & 3 \\ 0 & 4 \end{pmatrix}$

Find the values of p and q such that $\mathbf{A} = p\mathbf{A}^2 + q\mathbf{I}$. **(3 marks)**

A **5** Demonstrate that the following matrices satisfy their own characteristic equations.

a $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & -1 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 7 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{pmatrix}$

E **6** Given that

$$\mathbf{M} = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 0 & -1 \\ 3 & 2 & 0 \end{pmatrix}$$

a show that the characteristic equation of **M** can be written as $\lambda^3 = \lambda^2 + 9\lambda - 6$. **(3 marks)**

b Hence show that $\mathbf{M}^4 = 10\mathbf{M}^2 + 3\mathbf{M} - 6\mathbf{I}$. **(3 marks)**

E **7** Given that

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 0 & 4 \\ 4 & -1 & 3 \end{pmatrix}$$

a find the characteristic equation of **A**. **(3 marks)**

b Show that $\mathbf{A}^2 - 2\mathbf{A} - \mathbf{I} = 20\mathbf{A}^{-1}$. **(3 marks)**

c Hence find $\mathbf{A}^{-1}$. **(3 marks)**

E/P **8** $\mathbf{M} = \begin{pmatrix} -3 & 2 & -1 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{pmatrix}$

Find the values of a, b and c such that $\mathbf{M} = a\mathbf{M}^3 + b\mathbf{M}^2 + c\mathbf{I}$. **(4 marks)**

Challenge

Show that any 2 × 2 matrix of the form $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies its own characteristic equation and hence prove the Cayley–Hamilton theorem for 2 × 2 matrices.

Problem-solving

You can change the scalar 0 into the zero matrix **0** by multiplying by the identity matrix: $0\mathbf{I} = \mathbf{0}$.

Mixed exercise 5

(E) **1** The matrix $\mathbf{M} = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}$ has eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -15$.

a For each eigenvalue, find a corresponding eigenvector. **(4 marks)**

b Find a matrix **P** such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP} = \begin{pmatrix} 5 & 0 \\ 0 & -15 \end{pmatrix}$. **(3 marks)**

(E) **2** A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is represented by the matrix

$$\mathrm{A} = \begin{pmatrix} -5 & 8 \\ 3 & -7 \end{pmatrix}$$

a Find the eigenvalues of **A**. **(3 marks)**

b Find Cartesian equations of the two lines passing through the origin which are invariant under T. **(3 marks)**

(E/P) **3** The matrix $\mathbf{A} = \begin{pmatrix} 4 & k \\ 2 & -2 \end{pmatrix}$ has a repeated eigenvalue.

a Find the value of k. **(4 marks)**

b Hence find any eigenvectors for the matrix **A**. **(3 marks)**

The matrix represents a transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

c Find the Cartesian equation of any lines passing through the origin that are invariant under the transformation T. **(2 marks)**

(E/P) **4** The matrix $\mathbf{M} = \begin{pmatrix} a & a \\ 2 & 1 \end{pmatrix}$ has complex eigenvalues.

a Find the set of possible values of a. **(6 marks)**

b Given that $a = -1$, find the eigenvectors of **M**. **(4 marks)**

The matrix represents a transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

c Explain why there are no invariant lines under the transformation T. **(1 mark)**

(E) **5** The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}$$

a Show that 2 is an eigenvalue of **A** and find a corresponding eigenvector. **(4 marks)**

Given that the other eigenvalue of **A** is 1,

b find a matrix **P** and a diagonal matrix **D** such that $\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP}$. **(4 marks)**

(E) **6** The two sequences x_n and y_n satisfy the recurrence relations

$$x_{n+1} = 2x_n - y_n,\ x_1 = 2$$
$$y_{n+1} = 4x_n - 3y_n,\ y_1 = 3 \qquad n \geqslant 1$$

These recurrence relations can be written in matrix form as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad n \geqslant 1$$

where $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$.

a Find the eigenvalues and corresponding eigenvectors of **A**. **(5 marks)**

b Hence write down matrices **P** and **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$ where **D** is a diagonal matrix. **(2 marks)**

New sequences u_n and v_n can be formed from x_n and y_n using the transformation

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \mathbf{P}^{-1}\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

c Show that $\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \mathbf{D}\begin{pmatrix} u_n \\ v_n \end{pmatrix}$. **(2 marks)**

d Hence find closed form expressions for the original sequences x_n and y_n. **(5 marks)**

E/P **7** $\mathbf{A} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

a Find a matrix **P** and a diagonal matrix **D** such that $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D}$. **(5 marks)**

b Hence, or otherwise, find $\mathbf{A}^{50}$. **(3 marks)**

E **8** Given that

$$\mathbf{A} = \begin{pmatrix} 4 & 5 \\ -1 & 2 \end{pmatrix}$$

a Find the characteristic equation of **A**. **(2 marks)**

b Hence show that $\mathbf{A}^3 = 23\mathbf{A} - 78\mathbf{I}$. **(3 marks)**

E/P **9** $\mathbf{A} = \begin{pmatrix} 7 & 1 \\ -1 & 2 \end{pmatrix}$

Find the values of p and q such that $\mathbf{A} = p\mathbf{A}^2 + q\mathbf{I}$. **(3 marks)**

A E **10** Given that 1 is an eigenvalue of the matrix

$$\begin{pmatrix} 3 & 1 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

a find a corresponding eigenvector **(2 marks)**

b find the other eigenvalues of the matrix. **(3 marks)**

E **11** $\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 1 & -1 \\ -6 & -5 & -3 \end{pmatrix}$

a Find the eigenvalues of matrix **A** and hence find a set of eigenvectors. **(7 marks)**

Matrix **A** represents a transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

b Explain why every linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ must have at least one invariant line. **(1 mark)**

c Find the vector equations of the invariant lines under T. **(3 marks)**

A E **12** $\mathbf{A} = \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

a Show that 4 is an eigenvalue of **A** and find the other two eigenvalues. **(4 marks)**

b Find the corresponding eigenvectors of **A**. **(4 marks)**

Matrix **A** represents a transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$.

c Write down the vector equations of any invariant lines under T. **(2 marks)**

13 The matrix **M** is given by

E $$\mathbf{M} = \begin{pmatrix} 4 & 1 & -1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

a Show that −2 is an eigenvalue of **M** and find the other two eigenvalues. **(4 marks)**

b For each of the eigenvalues, find a corresponding eigenvector. **(4 marks)**

c Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{MP}$ is a diagonal matrix and write down the diagonal matrix **D**. **(3 marks)**

E **14** $\mathbf{A} = \begin{pmatrix} 3 & 4 & -4 \\ 4 & 5 & 0 \\ -4 & 0 & 1 \end{pmatrix}$

a Show that 3 is an eigenvalue of **A** and find the other two eigenvalues. **(4 marks)**

b Find an eigenvector corresponding to the eigenvalue 3. **(2 marks)**

Given that the vectors $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ are eigenvectors corresponding to the other two eigenvalues,

c find a matrix **P** such that $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$ is a diagonal matrix. **(3 marks)**

E **15** $\mathbf{A} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$

a Show that $\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ are eigenvectors of **A**, giving their corresponding eigenvalues. **(4 marks)**

b Given that 6 is the third eigenvalue of **A**, find a corresponding eigenvector. **(2 marks)**

c Hence write down a matrix such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. **(3 marks)**

16 **a** Show that for all values of the constant α, an eigenvalue of the matrix **A** is 1, where

E $$\mathbf{A} = \begin{pmatrix} \alpha & 0 & 2 \\ 4 & 3 & 0 \\ -2 & -1 & 1 \end{pmatrix}$$ **(3 marks)**

An eigenvector of the matrix **A** is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and the corresponding eigenvalue is β $(\beta \neq 1)$.

A

b Find the value of α and the value of β. **(4 marks)**

c For your value of α, find the third eigenvalue of **A**. **(2 marks)**

E **17** $\mathbf{M} = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

a Show that the matrix **M** has only two distinct eigenvalues. **(4 marks)**

b Find a set of eigenvectors for the matrix. **(4 marks)**

E **18** **a** Determine the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & -3 & 6 \\ 0 & 2 & -8 \\ 0 & 0 & -2 \end{pmatrix}$$ **(4 marks)**

b Show that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of **A**. **(2 marks)**

$\mathbf{B} = \begin{pmatrix} 7 & -6 & 2 \\ 1 & 2 & 3 \\ 1 & -3 & 2 \end{pmatrix}$

c Show that $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector of **B** and write down the corresponding eigenvalue. **(3 marks)**

d Hence, or otherwise, write down an eigenvector of the matrix **AB**, and state the corresponding eigenvalue. **(2 marks)**

E **19** Given that

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 3 & -3 \end{pmatrix}$$

a Find the characteristic equation of **A**. **(3 marks)**

b Show that $\mathbf{A}^2 + 2\mathbf{A} + 11\mathbf{I} = -6\mathbf{A}^{-1}$. **(3 marks)**

c Hence find $\mathbf{A}^{-1}$. **(3 marks)**

Challenge

The **trace** (tr) of a matrix is defined as the sum of the elements along the leading diagonal.

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

a Show that tr(**AB**) = tr(**BA**).

b Hence prove that, if there exists a non-singular matrix **P** such that $\mathbf{P}^{-1}\mathbf{MP} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, then the trace of matrix **M** is equal to $p + q$.

Summary of key points

1 An **eigenvector** of a matrix **A** is a non-zero column vector **x** which satisfies the equation $\mathbf{Ax} = \lambda\mathbf{x}$, where λ is a scalar.

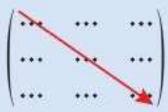
The value of the scalar λ is the eigenvalue of the matrix corresponding to the eigenvector **x**.

2 If **x** is an eigenvector of a matrix **M** representing a linear transformation, then the straight line that passes through the origin in the direction of **x** is an invariant line under that transformation.

3 The equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is called the **characteristic equation** of **A**. The solutions to the characteristic equation are the eigenvalues of **A**.

4 If $\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenvector of a matrix **A**, then the unit vector $\hat{\mathbf{a}} = \begin{pmatrix} \frac{a}{|\mathbf{a}|} \\ \frac{b}{|\mathbf{a}|} \end{pmatrix}$ is a **normalised eigenvector** of **A**.

5 A **diagonal matrix** is a square matrix in which all of the elements which are not on the diagonal from the top left to the bottom right of the matrix are zero. The diagonal from the top left to the bottom right of the matrix is called the **leading diagonal**.



The general 2 × 2 diagonal matrix is $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

6 To reduce a given matrix **A** to diagonal form, use the following procedure:

- Find the eigenvalues and eigenvectors of **A**.
- Form a matrix **P** which consists of the eigenvectors of **A**.
- Find $\mathbf{P}^{-1}$.
- A diagonal matrix **D** is given by $\mathbf{P}^{-1}\mathbf{AP}$.

7 When you reduce a matrix **A** to a diagonal matrix **D**, the elements on the diagonal are the eigenvalues of **A**.

8 A matrix, **A**, is **symmetric** if $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$. The elements of a symmetric matrix are symmetric with respect to the leading diagonal.

9 The procedure for **orthogonal diagonalisation** of a symmetric matrix **A** is:

- Find the normalised eigenvectors of **A**.
- Form a matrix **P** which consists of the normalised eigenvectors of **A**.
- Write down $\mathbf{P}^{\mathrm{T}}$.
- A diagonal matrix **D** is given by $\mathbf{P}^{\mathrm{T}}\mathbf{AP}$.

10 For a diagonal matrix $\mathbf{D} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, $\mathbf{D}^k = \begin{pmatrix} a^k & 0 \\ 0 & d^k \end{pmatrix}$.

11 The **Cayley–Hamilton theorem** states that every square matrix **M** satisfies its own characteristic equation.