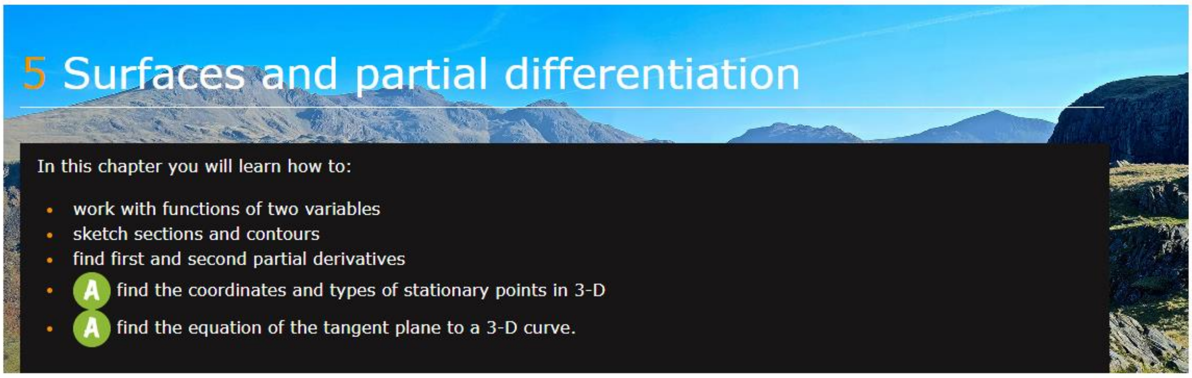


5 Surfaces and partial differentiation

In this chapter you will learn how to:

- work with functions of two variables
- sketch sections and contours
- find first and second partial derivatives
- A find the coordinates and types of stationary points in 3-D
- A find the equation of the tangent plane to a 3-D curve.

Section 1: Three-dimensional (3-D) surfaces

Key point 5.1

In three dimensions (3-D) a **surface** is described by the form:
$z = \mathrm{f}(x, y)$ or $\mathrm{g}(x, y, z) = c$, where c is a constant,
i.e. the height z above the $x-y$ plane is found directly or indirectly from the given equation. Conventionally, the $x-y$ plane is horizontal and the z-axis is vertical (using the right-hand system).
Plotting all possible points gives the surface.
Sections: these are cross-sections of the surface for defined values of x or y:
i.e. graphs of $z = \mathrm{f}(a, y)$ or $z = \mathrm{f}(x, b)$ for specific values of a or b.
Contours: these are effectively a plan view of the surface looking from above the $x-y$ plane for defined values of z (as on an Ordnance Survey map):
i.e. graphs of $c = \mathrm{f}(x, y)$ for specific values of c.
A **Multi-variable functions**: it is possible to have more variables such as $w = \mathrm{f}(x, y, z)$, where w is a variable.

EXERCISE 5A

In this exercise you are encouraged to use graph plotting software or a graphical calculator.

1 For the surface $z = x^2 + y^2$:

a on separate diagrams, sketch the sections **i** $z = x^2$ and **ii** $z = y^2$

b draw the contours for $x^2 + y^2 = c$ for c taking values $1, 4, 9$ and 16.

2 For the surface $z = x^2 + 9y^2$:

a on separate diagrams, sketch the sections **i** for $y = 0$ and $y = 1$ **ii** for $x = 0$ and $x = 2$

b draw the contours for $x^2 + 9y^2 = c$ for c taking values $0, 1, 4, 9$ and 16.

3 A surface has equation $z = x^2 - y^2$.

a Sketch the section with $y = 2$ and state the coordinates of the points of intersection of this section with the plane $z = 0$.

b Sketch the section with $x = 4$ and state the coordinates of the points of intersection of this section with the plane $z = 7$.

c Draw the contours for $x^2 - y^2 = c$ for c taking values $0, 1, 4, 9$ and 16.

4 A surface has equation $z = (x + y)^2$.

a Sketch the sections with $y = 0$, $y = 1$ and $y = -3$. Where the sections intersect the $x-y$ plane find the coordinates of the points of intersection.

b Draw the contours for $(x + y)^2 = c$ for c taking values $0, 1, 4, 9$ and 16.

5 A surface has equation $z=(x-y)^2$.

a Sketch the sections with $x=0$, $x=2$ and $x=-1$. Where the sections intersect the plane $z=4$ find the coordinates of the points of intersection.

b Draw the contours for $(x-y)^2=c$ for c taking values $0, 1, 4, 9$ and 16.

6 A surface has equation $z=(x^2-y^2)^2$.

a Determine whether the point $(3, 1, 64)$ lies on the surface.

b Find whether the point $(5, 4, 80)$ is on, above or below the surface.

c Draw the contours for $(x^2-y^2)^2=c$ for c taking values $0, 1, 4, 9$ and 16.

7 A surface S has equation $z=25-(x^2+y^2)$.

a On separate diagrams, sketch the sections **i** for $y=0$ and **ii** for $x=4$.

b Find a point on S which lies in the $x-y$ plane.

c Draw the contours for $25-(x^2+y^2)=c$ for c taking values $0, 1, 4, 9$ and 16.

8 For the surface $z=\sqrt{x}+\sqrt{y}$:

a on separate diagrams, sketch the sections **i** $z=\sqrt{x}$ and **ii** $z=\sqrt{y}$

b draw the contours for $\sqrt{x}+\sqrt{y}=c$ for c taking values $1, 4$ and 9.

9 For the surface $z=x^2y^2$:

a on separate diagrams, sketch the sections **i** for $y=1$ and **ii** for $x=1$

b draw the contours for $x^2y^2=c$ for c taking values $1, 4, 9$ and 16.

10 For the surface $z=x^2+y^3$:

a on separate diagrams, sketch the sections **i** for $y=3$ and **ii** for $x=4$

b verify that the two sections in **a** intersect where $x=-4$ and $y=3$ and state the coordinates of the point on the surface of this intersection

c draw the contours for $x^2+y^3=c$ for c taking values $1, 4, 9$ and 16.

11 For the surface $z=e^{x+y}$:

a on separate diagrams, sketch the sections **i** for $y=0$ and **ii** for $x=0$

b draw the contours for $e^{x+y}=c$ for c taking values $1, 4, 9$ and 16.

12 A surface S has equation $z=x^2+xy+y$.

a Draw sections of S for $y=0$, $y=-2$, $y=2$.

b State which of the sections in **a** intersect the plane $z=0$.

13 For the surface $z=xe^y$:

a on separate diagrams, sketch the sections **i** for $y=0$ and **ii** for $x=1$

b draw the contours for $xe^y=c$ for c taking values $1, 3$ and 5.

14 For the surface $z=\ln(xy)$:

a on separate diagrams, sketch the sections **i** for $y=1$ and **ii** for $x=1$

b draw the contours for $\ln(xy)=c$ for c taking values $0, 1$ and 4.

15 A surface S has equation $z=x^3+3xy-y^2$.

a Sketch the sections of S when $x=3$ and $x=-3$.

b State which of the sections in **a** intersect the plane $z=0$.

c Use a graph plotting package to sketch the contour of S when $z=0$.

16 The surface S of a three-dimensional object has equation $z=x^3+3x^2y+y^2+3$.

a State the equation of the section of S for which $y=-1$ and sketch this section.

b Find the coordinates of the points where the section intersects the plane $z=0$.

c Find the coordinates of the turning points on this section of S.

17 For the surface $z=\sin(x)+\sin(y)$:

a on separate diagrams, sketch the sections when **i** $y=\frac{\pi}{2}$ and **ii** $x=\pi$

b draw the contours for $\sin(x)+\sin(y)=c$ for $c=0$

c use a graph plotter to draw the contours for $\sin(x)+\sin(y)=0.5$ and $\sin(x)+\sin(y)=1$.

18 For the surface $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=25$:

a on separate diagrams, sketch the sections **i** $\frac{x^2}{4}+\frac{z^2}{16}=25$ and **ii** $\frac{y^2}{9}+\frac{z^2}{16}=25$

b draw the contours for $\frac{x^2}{4}+\frac{y^2}{9}+\frac{c^2}{16}=25$ for c taking values $4, 12$ and 16.

19 For the surface $z^2=\frac{x^2}{16}+\frac{y^2}{9}$:

a on separate diagrams, sketch the sections **i** for $y=0$ and **ii** for $x=0$

b draw the contours for $c^2=\frac{x^2}{16}+\frac{y^2}{9}$ for c taking values $2, 3$ and 4.

20 For the surface, $z^2=\frac{x^2}{4}+\frac{y^2}{9}-1$:

a on separate diagrams, sketch the sections **i** for $y=0$ and **ii** for $x=0$

b draw the contours for $c^2=\frac{x^2}{4}+\frac{y^2}{9}-1$ for c taking values $2, 3$ and 4.

21 For the surface $z^2=\frac{x^2}{4}+\frac{y^2}{9}+1$:

a on separate diagrams, sketch the sections with **i** $y=0$, $y=3$ and $y=6$ and **ii** $x=0$, $x=1$ and $x=2$

b draw the contours for $c^2=\frac{x^2}{4}+\frac{y^2}{9}+1$ for c taking values $0, 1, 4, 9, 16$.

Section 2: Partial differentiation

Key point 5.2

If a 3-D surface has equation $z=\text{f}=\text{f}(x,y)$, then:

$\text{f}_x=\frac{\partial \text{f}}{\partial x} \Rightarrow$ differentiate f with respect to x, assuming y is constant

which gives the rate of change on the surface of z as x changes

$\text{f}_y=\frac{\partial \text{f}}{\partial y} \Rightarrow$ differentiate f with respect to y, assuming x is constant

$\text{f}_{xx}=\frac{\partial}{\partial x}\left(\frac{\partial \text{f}}{\partial x}\right)=\frac{\partial^2 \text{f}}{\partial x^2} \Rightarrow$ differentiate f_x with respect to x, assuming y is constant

$\text{f}_{yy}=\frac{\partial}{\partial y}\left(\frac{\partial \text{f}}{\partial y}\right)=\frac{\partial^2 \text{f}}{\partial y^2} \Rightarrow$ differentiate f_y with respect to y, assuming x is constant

$\text{f}_{xy}=\frac{\partial}{\partial x}\left(\frac{\partial \text{f}}{\partial y}\right)=\frac{\partial^2 \text{f}}{\partial x \partial y} \Rightarrow$ differentiate f_y with respect to x, now assuming y is constant

$\text{f}_{yx}=\frac{\partial}{\partial y}\left(\frac{\partial \text{f}}{\partial x}\right)=\frac{\partial^2 \text{f}}{\partial y \partial x} \Rightarrow$ differentiate f_x with respect to y, now assuming x is constant.

Note: the **mixed derivative theorem** for most well-behaved, continuous functions, states that:

$\text{f}_{xy}=\text{f}_{yx}$

EXERCISE 5B

For questions 1 to 15, find **a** f_x, **b** f_y, **c** f_{xx} and **d** f_{yy}. Also show that $\text{f}_{xy}=\text{f}_{yx}$.

1 $\text{f}=x^2+y^2$

2 $\text{f}=x^2y^2$

3 $\text{f}=x^3+y^3+3x^2y^2$

4 $\text{f}=x^2y^2-x^4+y^4$

5 $\text{f}=(x^2+y^2)^2$

A 6 $f = (x^2 - y^2)^{\frac{1}{2}}$

A 7 $f = (2x + 2y - x^2 - y^2)^{\frac{1}{2}}$

A 8 $f = \sqrt{2x+1}\sqrt[3]{2y+2}$

A 9 $f = \frac{x^2+2}{y^2+2}$

A 10 $f = e^{x+y}$

11 $f = \ln(2xy)$

12 $f = x^2y^2\ln(xy)$

13 $f = \cos(x) + \cos(y)$

14 $f = \cos(x)\sin(y)$

A 15 $f = e^{\sin x - \cos y}$

16 Given that $2z - 4 = 3x^2 + 5y^2 + 4xy$, find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ and show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

17 If $f = x^3 + y^3 + 3xy$, show that, at the point $(-1, -1, 1)$:

a $f_x = 0$

b $f_y = 0$

c $f_{xx} < 0$

d $f_{yy} < 0$

e $f_{xx}f_{yy} - (f_{xy})^2 > 0$.

18 Show, for the point $(1, 1, -2)$ on the curve $z = x^4 + y^4 - 2x^2 - 2y^2$, that:

a $\frac{\partial^2 z}{\partial x^2} > 0$

b $\frac{\partial^2 z}{\partial x^2} > 0$

c $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 > 0$

19 If $f = xy\ln(x - y)$, show that, at the point $(1, 0, 0)$:

a $f_x = 0$

b $f_y = 0$

c $f_{xx}f_{yy} - (f_{xy})^2 < 0$

20 Given that $f = x^3 + y^3 + z^3 + 3x^2y + 3y^2z + 3xz^2 + x^2y^2z^2$, find:

a f_x

b f_y

c f_z

d f_{xyz}

Section 3: Stationary points

A **EXERCISE 5C**

1 A surface S has equation $z = x^2 + y^2$.

Find the coordinates and nature of the stationary point and show that it is a minimum.

2 Find the coordinates and nature of the stationary point on the surface $z = 4 - (x^2 + y^2)$.

3 A surface S is defined by $z = x^2 - y^2$. Show that the only stationary point is a saddle point.

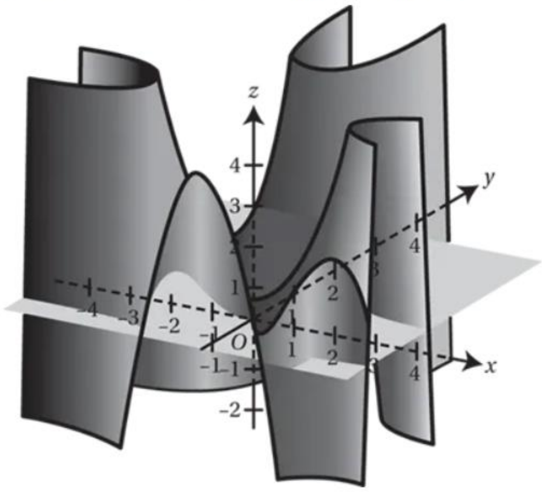
4 Find the coordinates and nature of the stationary point on $z = x^3 + y^3 - 6xy$.

5 Find the coordinates of the four stationary points on $z = 3x^2y - 3x^2 - 6y^2 + y^3$ and determine whether each stationary point is a maximum, a minimum or a saddle point.

6 Show that $f(x, y) = x^4 - 2x^2 + y^2$ has three stationary points.

Find the coordinates of each stationary point and determine whether each is a maximum, a minimum or a saddle point.

7 The diagram shows a stand used in a museum display.



The surface of the stand is given by the equation $z = x^2 + y^2 - x^2y^2$. Show that the stand has four saddle points and one minimum stationary point.

Find the coordinates of all these points.

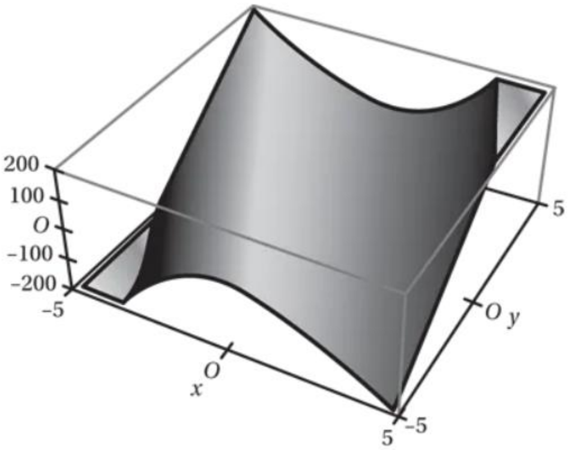
8 A modern building has a roof surface with equation:

$z = 2x^3 - 3x^2 - 12x + y^3 - 3y$.

a Find the position of the highest point and the lowest point on the roof.

b The roof has two supports placed at the saddle points. Find the coordinates of the saddle points.

9 A design department of the manufacturer of corn snacks produces a template of the following shape:



This surface has the equation: $z = x^3 + 3x^2y - 3y - 5$ and it has two stationary points.

Show that the coordinates of these are $(-1, 0.5, -6)$ and $(1, -0.5, -4)$ and determine whether each point is a maximum, a minimum or a saddle point.

10 Show that $f(x, y) = x^4 - 2x^2 + y^4 - 2y^2$ has nine stationary points.

Verify that four of these are saddle points.

For the other five points, determine whether each is a maximum or a minimum stationary point.

11 Find the coordinates of the stationary point on $z = e^{xy}$ and prove that it is a saddle point.

12 Given that $z = e^{x^2+y^2}$, show that there is a minimum stationary point at $(0, 0, 1)$.

13 A roof light has a surface given by the equation $z = e^{-(x^2+y^2)}$, for $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant= 1$, where all lengths are in metres.

a Show that the roof light has a maximum height of $1\,\mathrm{m}$ and that this occurs at its centre.

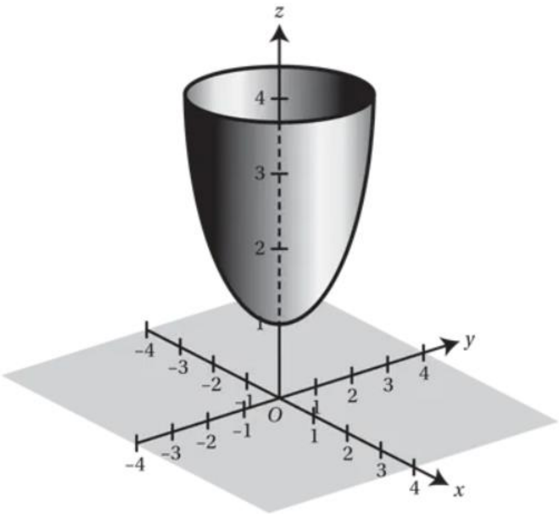
b Sketch the section of S for which $y = 0$.

c Draw a contour map for S for z values $0.5, 0.25, 0.1$ and 0.01.

14 Standing waves in a water tank can be modelled by the equation $z = \cos(x) + \sin(y)$. Show that the surface of these waves bounded by $0 \leqslant x \leqslant \pi$ and $0 \leqslant y \leqslant \pi$ has two stationary points. Find the coordinates of these points and determine whether each is a maximum, a minimum or a saddle point.

15 Find the coordinates of the stationary points on $z = xy + \ln(x^2 + y^2)$ and determine whether each is a maximum, a minimum or a saddle point.

16 To reduce drag the nose cone of a Formula 1 racing car looks like the one in the diagram.



Given that the equation of this surface is $z = x^2 + y^2 + e^{xy}$, show that there is a minimum stationary point at $(0, 0, 1)$.

17 Show that $z = x^3 + y^3 - 3x - 3y$ has two saddle points, one maximum stationary point and one minimum stationary point.

Find the coordinates of all these points.

18 A surface S has equation $z = e^{\sin x} + e^{\cos y}$.

a Show that the (x, y) coordinates $\left(-\frac{\pi}{2}, 0\right)$, $\left(-\frac{\pi}{2}, \pi\right)$, $\left(\frac{\pi}{2}, 0\right)$, $\left(\frac{\pi}{2}, \pi\right)$ give stationary points on S.

b Determine whether each is a maximum, a minimum or a saddle point.

19 Show that $(0, 0, 0)$ is a stationary point on $z = x^2 + y^2 - 2kxy$ $(k^2 \neq 1)$.

Determine the type of stationary point depending on the values of k.

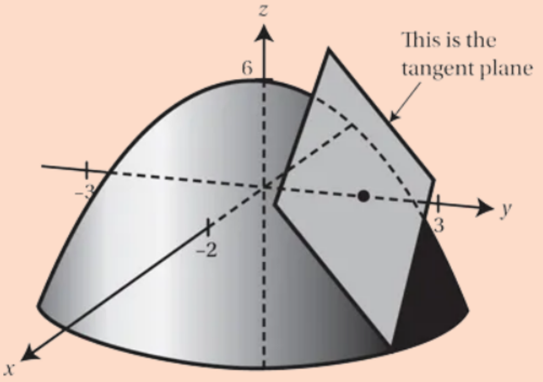
20 A surface C has equation $z = \cos(x)\cos(y)$ for $-2\pi < x < 2\pi$ and $-2\pi < y < 2\pi$.

Show that C has maximum turning points at $(0, 0)$, $(\pm\pi, \pm\pi)$ and $(\pm 2\pi, \pm 2\pi)$ and minimum turning points at $(0, \pm\pi)$, $(\pm 2\pi, \pm\pi)$, $(\pm\pi, 0)$ and $(\pm\pi, \pm 2\pi)$.

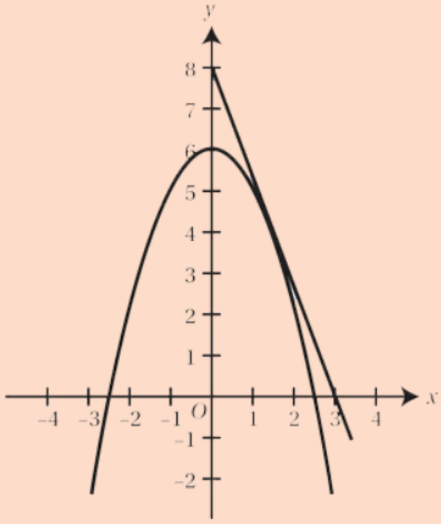
Section 4: Tangent planes

Key point 5.4

For a 3-D surface with equation $z = f(x, y)$, the equation of the **tangent plane** to the surface at the point $(a, b, f(a, b))$ is:
$z = f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b)$



The tangent plane is a 3-D version of a tangent to a curve.



EXERCISE 5D

1 Show that the equation of the tangent plane to $z = x^2 + y^2$ at the point $(3, 2, 13)$ is $z = 6x + 4y - 13$.

2 A surface S has equation $z = x^2 - y^2$. Find the equation of the tangent plane at the point where $x = 2$ and $y = 3$.

3 Find the equation of the tangent plane to $z = x^2 + y^2 + 3x^2y$ at the point $(4, -1, -31)$.

4 Show that the point $P(2, 1, 12)$ lies on the surface $z = x^3 + x^2 + 3 + y^2 - 2xy$.

Find the equation of the tangent plane at P.

5 Show that the equation of the tangent plane to $z = (x^2 + y^2)^2$ at the point $(2, -2, 64)$ is $z = 64x - 64y - 192$.

6 Find the equation of the tangent plane to $z = 3x^2 + 2y^3 - xy + 6$ at the point where $x = -1$ and $y = 1$.

7 Find the equation of the tangent plane to $z = x^3 + y^3 + x^2y^2 - 3x - 5y$ at the point where $x = 2$ and $y = 1$.

8 Find the equation of the plane that is tangential to $z = \frac{x}{y}$ at the point $(4, 2, 2)$.

9 Find the equation of the tangent plane to $z = \frac{x+1}{y-2}$ at the point where $x = 2$ and $y = 4$.

10 A surface has equation $z = (x+1)^{\frac{1}{2}}(y+2)^{\frac{3}{2}}$. Find the equation of the plane that is tangential to S at the point $(3, 7, 54)$.

11 Show that the equation of the tangent plane to $z = e^{x^2+y^2}$ at the point where $x = 1$ and $y = 1$ is $z = e^2(2x + 2y - 3)$.

12 Find the equation of the tangent plane to $z = y^2e^{x^2}$ at the point where $x = 1$ and $y = 2$.

13 Show that the equation of the tangent plane to $z = \ln(x + y + 2xy)$ at the point $(4, 1, \ln 13)$ is $13z = 3x + 9y - 21 + 13\ln 13$.

14 Find the equation of the plane that is tangential to the surface $z = \cos(x) + \cos(y)$ at the point $\left(0, \frac{\pi}{2}, 1\right)$.

15 Given that a surface S has equation $z = \cos(x)\sin(y)$, find the equation of the plane that is tangential to S at the point where $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$.

16 Show that the equation of the plane that is tangential to the surface with equation $z = x^4 - 4x^2y^2 + y^4$ at the point $(2, 1)$ is $\mathbf{r} \bullet \begin{pmatrix} 16 \\ -28 \\ 1 \end{pmatrix} = 3$.

17 Show that the point $(1, 2, 2)$ lies on the plane that is tangential to $z = xy + e^x \ln y$ at $(0, 1, 1)$.

18 Determine whether the plane $\mathbf{r} \bullet \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 2$ is parallel to the plane that is tangential to the surface $z = \cos(xy) + x + y - 2$ at the point where $x = 0$ and $y = \pi$.

19 Find the equation of the tangent plane to $z = \tan(x)\tan(y)$ at the point where $x = 0$ and $y = \frac{\pi}{4}$.

20 A surface S has equation $x^2 + y^2 + z^2 = 50$.

a Show that the plane that is tangential to S at the point $(3, 4, 5)$ has equation $3x + 4y + 5z = 50$

b Sketch a section of S for which $y = 0$.

c Sketch a contour diagram for values of $z \in \{1, 3, 5, 7\}$.

Mixed practice 5

1 A curve has equation $z = 4x^2 + 9y^2$.

a Sketch sections:

i $z = 4x^2$

ii $z = 9y^2$.

b Sketch contours for $4x^2 + 9y^2 = c$, where $c = 1, 4, 9$.

2 Show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$ for:

a $z = \frac{x - y}{x + y}$ noting that $\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$

b $z = \tan^{-1}\frac{y}{x}$ noting that $\frac{\partial}{\partial x}(\tan^{-1}z) = \frac{1}{1 + z^2}\frac{\partial z}{\partial x}$

3 Find the equation of the tangent plane to $4z = 25 - 3x^2 - y^2$ at the point where $x = 2$ and $y = 1$.

4 Show that the curve $z = x^2 + 2y^2 + 2x - y$ has a minimum turning point at $(-1, 0.25, -1.125)$.

5 A surface S has equation $z = f(x, y)$ where $f(x, y) = x^3 - 3x^2y + 3y^2$.

Show that S has a saddle point at $(1, 0.5, 0.25)$.

6 Find the equation of the tangent plane to $z = xy^2$ at the point where $x = -5$ and $y = 1$.

7 It is given that $f(x, y) = e^{-(x^2+y^2)}$.

a Show that $f_{xy} = 4xye^{-(x^2+y^2)}$.

b Find the stationary point on the surface $z = f(x, y)$ and explain why it is a maximum, minimum or saddle point.

c The surface has sections $z = f(a, y)$, where a is a constant greater than zero. Find, in terms of a, the coordinates of the turning point of this section. Sketch this section.

8 A surface has equation $z = f(x, y)$, where $f(x, y) = x^2 + y^2 + \frac{2}{xy}$.

a Find the stationary points on the curve and determine whether each is a minimum, maximum or saddle point.

b Sketch a section for which $y = 1$.

9 An open-topped box has volume $1\,\text{m}^3$. Its base has dimensions $x \times y$ and its height is z.

a Find an expression for the surface area in terms of x and y.

b Use partial differentiation to prove that the surface area is a minimum when $x = 2^{\frac{1}{3}}$, $y = 2^{\frac{1}{3}}$ and $z = \frac{1}{2^{\frac{2}{3}}}$.

10 Find the equation of the tangent plane to $xy + yz + zx = 11$ at the point where $x = 1$ and $y = 2$, writing your answer in the form $f(x, y, z) = k$, where k is a constant.

11 Given that u and v are functions of x and y and that $ux = vy, u^2y = vx^2$, find $\frac{\partial u}{\partial x}$ and show that $\frac{\partial v}{\partial x} = \frac{2u^2y^2 + xv^2}{y(2uy^2 - 2vx^2)}$.