4 Recurrence relations

Objectives

After completing this chapter you should be able to:

- Use recurrence relations to describe sequences and model situations → pages 000–000
- Find solutions to first-order recurrence relations → pages 000–000
- Find solutions to second-order recurrence relations → pages 000–000
- Use mathematical induction to prove closed forms for recurrence relations → pages 000–000

In population modelling, the final population for one year becomes the starting population for the next year. You can model the population at the end of each year as a sequence and describe it using a recurrence relation. → Exercise 4B Q8

Prior knowledge check

1 A sequence of numbers is defined by the recurrence relation

$u_{k+1} = 3u_k$, with $u_0 = 2$

a Write down the first five terms of the sequence.

b Find an expression for u_n in terms of n.

← Pure Year 2, Chapter 3

2 A sequence of numbers is defined for all $n \in \mathbb{N}$ by

$u_{n+1} = au_n + b$, with $u_1 = 3$

Given that $u_2 = 5$ and $u_3 = 9$, find the values of a and b. ← Pure Year 2, Chapter 3

3 Prove by induction that for all positive integers n, $\sum_{r=1}^{n}(2r - 1) = n^2$

← Core Pure Book 1, Chapter 8

4.1 Forming recurrence relations

You can model many real-life situations using recurrence relations.

For example, suppose that you have £500 in a savings account that pays 0.5% interest every month. Each month, you add another £100 to the savings account.

Links A recurrence relation describes each term of a sequence in terms of the previous term or terms. ← Pure Year 2, Section 3.7

You can use this information to formulate a recurrence relation that describes the amount in the account at the end of each month.

Let u_m be the amount in pounds in the account after m months. The next month, $m + 1$, you will have the original amount, u_m, plus the interest, $0.005u_m$, plus the additional £100 you add every month. This generates the recurrence relation

$$u_{m+1} = u_m + 0.005u_m + 100, \text{ with } u_0 = 500$$

You need to give the initial amount in the account to fully define the sequence. This is sometimes called an **initial condition** for the recurrence relation.

Notation This is an example of a **first-order recurrence relation,** as u_{m+1} is given in terms of **one previous term**, u_m. → Section 4.2

Example 1

You owe £500 on a credit card that charges 1.5% interest each month. You decide to make no new charges and you pay off £50 each month. Formulate a recurrence relation that describes the balance remaining on the credit card after n months.

Let u_n be the amount in pounds owed after n months.

Define the terms and give any relevant units.

During a month, the interest is $0.015u_n$ and you pay off £50.

*The interest is **added** to the balance and the amount you pay off is **subtracted** from it.*

$u_{n+1} = u_n + 0.015u_n - 50 = 1.015u_n - 50$, with $u_0 = 500$

*Remember to give the **initial condition**.*

Example 2

The deer population of a county was observed to be 1200 in a given year. The population is modelled to increase at a rate of 15% each year. Let d_n be the population of deer n years later. Explain why the deer population is modelled by the recurrence relation

$$d_n = 1.15d_{n-1}, \text{ with } d_0 = 1200$$

After $n - 1$ years the population is d_{n-1}

This is increased by 15%, so the population after n years is $d_{n-1} + 0.15d_{n-1} = 1.15d_{n-1}$

The initial population is 1200, so $d_0 = 1200$

Explain the recurrence relation in the context of the question.

Example 3

A population of bacteria has initial size 200. After one hour, the population has reached 220. The population grows in such a way that the rate of growth doubles each hour. Write a recurrence relation to describe the number of bacteria, b_n, after n hours.

$b_0 = 200$ and $b_1 = 220$
The increase from time $n - 1$ to n is $b_n - b_{n-1}$, and the increase from time $n - 2$ to $n - 1$ is $b_{n-1} - b_{n-2}$
So $b_n - b_{n-1} = 2(b_{n-1} - b_{n-2})$
$b_n = 3b_{n-1} - 2b_{n-2}$, with $b_0 = 200$, $b_1 = 200$

The **rate** of growth doubles each hour. So the increase from time $n - 1$ to n will be double the increase from time $n - 2$ to $n - 1$.

Notation This is an example of a **second-order recurrence relation**, as b_{n+1} is given in terms of **two previous terms**, b_{n-1} and b_{n-2}. You need **two initial conditions** to define the sequence, given here in terms of b_0 and b_1. → Section 4.3

If you know the general term of a sequence in the form $u_n = \text{f}(n)$, you can verify that it satisfies a given recurrence relation by substitution.

Example 4

A sequence has the general term $u_n = 3n - 1$. Verify that the sequence satisfies the recurrence relation $u_n = 3 + u_{n-1}$.

$u_n = 3n - 1$, so $u_{n-1} = 3(n - 1) - 1 = 3n - 4$
Substituting into the RHS of the recurrence relation,
$3 + u_{n-1} = 3 + (3n - 4)$
$= 3n - 1$
$= u_n$ as required

Watch out This is not the only general term that satisfies this recurrence relation. Any general term of the form $u_n = 3n + k$, where k is a constant, will also satisfy the recurrence relation. If you want to prove that a general term satisfies a given recurrence relation with an initial condition, you can do this using mathematical induction. → Section 4.4

Example 5

A sequence has the general term $u_n = 2 \times 3^{n-1}$. Verify that the sequence satisfies the recurrence relation $u_n = 3 + u_{n-1}$.

$u_{n-1} = 2 \times 3^{(n-1)-1} = 2 \times 3^{n-2}$
Substituting into the RHS of the recurrence relation,
$3u_{n-1} = 3(2 \times 3^{n-2}) = 2 \times 3^{n-1} = u_n$

Notation $u_n = 3u_{n-1}$ is the **recursive form** of the sequence. $u_n = 2 \times 3^{n-1}$ is the **solution**, or the **closed form** of the sequence. It is also sometimes called the **explicit form** of the sequence. → Section 4.2

Exercise 4A

1 The value of an endowment policy increases at a rate of 5% per annum.
The initial value of the policy is £7000.

a Write down a recurrence relation for the value of the policy after n years.

b Calculate the value of the policy after 4 years.

Hint Remember to include an initial condition in your answer to part **a**.

2 A patient is injected with 156 ml of a drug. Every 8 hours, 22% of the drug passes out of his bloodstream, and a further 25 ml dose of the drug is administered. After $8n$ hours, the amount of the drug in the patient's bloodstream is d_n ml.

a Find an expression for d_n in terms of d_{n-1}, and write down the value of d_0.

b Calculate, to the nearest millilitre, the amount of drug in the patient's bloodstream after 24 hours.

(E) **3** Kandace takes out a personal loan of £5000 to buy a car. The interest rate on the loan is 0.5% per month. Interest is calculated and added to the loan balance at the end of each month.
At the end of each month, Kandace makes a monthly payment of £200, which is deducted from the balance of the loan. The balance in pounds at the end of the nth month is given by b_n.
Explain why $b_n = kb_{n-1} - 200$, with $b_0 = 5000$, and find the value of the constant k. **(3 marks)**

(E) **4** At the time a census is taken, the population of a country is 12.5 million. The annual birth rate is 4% and the annual death rate is 3%. In addition, each year there is a net migration of 50 000 new immigrants into the country.
Write a recurrence relation for the population of the country n years after the census, P_n. **(3 marks)**

5 A sequence has general term $u_n = 5n + 2$. Verify that the sequence satisfies the recurrence relation $u_n = u_{n-1} + 5$.

6 A sequence has general term $u_n = 6 \times 2^n + 1$. Verify that the sequence satisfies the recurrence relation $u_n = 2u_{n-1} - 1$.

(P) **7** Consider the sequence given by $u_n = \sum_{i=1}^{n}(2i - 1)$

a Write down the first 4 terms of the sequence.

b Explain why the recurrence relation associated with this sequence is
$$u_{n+1} = u_n + 2n + 1, n \geqslant 1$$

c Verify that $u_n = n^2$ is a solution to this recurrence relation.

(E/P) **8** In January 2010, a small oil company produced 2000 barrels of oil and sold 1800 barrels of oil. Any remaining oil was stockpiled. From January 2010 onwards, the company increased its sales by 20 barrels per month, and increased its oil production by 1% each month.

a Find an expression for:

i the number of barrels produced by the well in the nth month

ii the number of barrels sold in the nth month. **(4 marks)**

At the beginning of January 2010, the oil company had no stockpiled oil.

b Find a recurrence relation for the total number of stockpiled barrels, s_n, at the end of the nth month. **(3 marks)**

E/P **9** There are n people at a gathering. Each person shakes hands with everybody else exactly once. Let $h(n)$ be the number of handshakes that occur.

a Explain why $h(1) = 0$. **(1 mark)**

b Find a recurrence relation for $h(n + 1)$ in terms of $h(n)$. **(2 marks)**

A **10** Generate the first six terms of each of the following sequences:

a $u_n = 2u_{n-1} + 3u_{n-2}$, with $u_0 = 1$ and $u_1 = 1$

b $u_n = u_{n-1} - 2u_{n-2}$, with $u_0 = 1$ and $u_1 = 1$

c $u_n = u_{n-1} + u_{n-2} + 2n$, with $u_0 = 1$ and $u_1 = 1$

E **11** Assume that growth in a bacterial population has the following properties:

- At the beginning of every hour, each bacterium that lived in the previous hour divides into two new bacteria. During the hour, all bacteria that have lived for two hours die.
- At the beginning of the first hour, the population consists of 100 bacteria.

At the end of the nth hour there are B_n bacteria in the population.
Find a recurrence relation for B_n. **(3 marks)**

P **12** A sequence has nth term $u_n = (2 - n)2^{n+1}$
Verify that the sequence satisfies the recurrence relation $u_n = 4(u_{n-1} - u_{n-2})$.

E/P **13** A battery-operated kangaroo is able to make two kinds of jumps: small jumps of length 10 cm or large jumps of length 20 cm. The number of different ways in which the kangaroo can cover a distance of $10n$ cm is denoted by J_n.

a By writing down all possible combinations of jumps for a distance of 40 cm, show that $J_4 = 5$. **(2 marks)**

b Find a recurrence relation for J_n, stating the initial conditions. **(3 marks)**

c How many different ways can this kangaroo cover a distance of 80 cm? **(1 mark)**

E/P **14** A female rabbit is modelled as producing 2 surviving female offspring in its first year of life, and 6 in each subsequent year. A population initially has 4 female rabbits, all of whom are more than 1 year old.

a If F_n is the number of female rabbits in the population after n years, explain why F_n is modelled by the recurrence relation

$$F_n = 3F_{n-1} + 4F_{n-2}, \text{ with } F_0 = 4 \text{ and } F_1 = 28$$ **(3 marks)**

b Suggest a criticism of this model. **(1 mark)**

A E/P **15** Binary strings consist of 1s and 0s. There are 5 different binary strings of length 3 which **do not** contain consecutive 1s:

000, 001, 010, 100, 101

Let b_n represent the number binary strings of length n with no consecutive 1s.

a Find b_1 and b_2. **(1 mark)**

b Explain why b_n satisfies the recurrence relation $b_n = b_{n-1} + b_{n-2}$. **(3 marks)**

c Hence find b_7. **(1 mark)**

Hint For example, 011 is not allowable because it contains consecutive 1s.

4.2 Solving first-order recurrence relations

You need to be able to **solve** recurrence relations. This means finding a **closed form** for the terms in the sequence in the form $u_n = \mathrm{f}(n)$.

- **The order of a recurrence relation is the difference between the highest and lowest subscript in the relation.**
- **A first-order recurrence relation is one in which u_n can be given as a function of n and u_{n-1} only.**

Examples of first-order recurrence relations are:

$u_n = 2u_{n-1} + n$

$a_n = (n + 1)a_n - 1$

$P_{n+1} = 5P_n + 2n^2$

Here the subscripts given are $n + 1$ and n. This is still a first-order recurrence relation because the difference between them is 1.

In this section you will learn how to solve first-order **linear** recurrence relations.

- **A first-order linear recurrence relation can be written in the form $u_n = au_{n-1} + \mathrm{g}(n)$, where a is a real constant.**
  - **If $\mathrm{g}(n) = 0$, then the equation is homogeneous.**

You can sometimes find solutions to recurrence relations using a technique called **back substitution**.

Example 6

Find a closed form for the sequence

$a_n = 5a_{n-1}, n > 0; a_0 = 1$

$a_n = 5a_{n-1}$

$= 5 \times 5a_{n-2} = 5^2a_{n-2}$

$= 5^2 \times 5a_{n-3} = 5^3a_{n-3}$

$= \ldots = 5^na_0.$

$a_0 = 1$, so the closed form of this sequence is $a_n = 5^n$

$a_{n-1} = 5a_{n-2}$. Substitute this into the expression for a_n.

This is the geometric sequence 1, 5, 25, 125 …

Watch out When you find a closed form by this method, you need to subsequently **prove** it using mathematical induction. → Section 4.4

The recurrence relation in the example above is an example of a **homogeneous** recurrence relation. A first-order homogeneous linear recurrence relation can be written in the form $u_n = au_{n-1}$.

Using back substitution,

$$\begin{aligned} u_n &= au_{n-1} \\ &= a \times au_{n-2} = a^2u_{n-2} \\ &= a^2 \times au_{n-3} = a^3u_{n-3} \\ &\vdots \\ &= a^{n-1}u_1 \\ &= a^n u_0 \end{aligned}$$

- **The solution to the first-order homogeneous linear recurrence relation $u_n = au_{n-1}$ is given by $u_n = u_0a^n$ or $u_n = u_1a^{n-1}$.**

Unless you are told to prove a recurrence relation by induction, you can write down these solutions in your exam.

Example 7

Solve the recurrence relation $a_n = 2a_{n-1}$, $n \geqslant 1$, with $a_0 = 3$.

$a_n = a_0(2^n)$
$\quad = 3(2^n)$

This is a homogeneous linear first-order recurrence relation, so use the rule given above to write down the general solution.

It is useful to think of a **general solution** to the recurrence relation $u_n = au_{n-1}$ in the form $u_n = ca^n$, where c is an arbitrary constant. You can then use the initial conditions to find the value of c. This will give you a **particular solution**.

Links The process of finding general solutions (with arbitrary constants), and then using initial conditions to find particular solutions, is very similar to the process of solving a differential equation. You can think of a recurrence relation as a discrete version of a differential equation. ← Pure Year 2, Section 11.10

Example 8

Solve the recurrence relation $a_n = -3a_{n-1}$, $n \geqslant 1$, with $a_1 = 6$.

Method 1

$a_n = a_1(-3)^{n-1}$
$\quad = 6(-3)^{n-1}$

Use the form of the solution $u_n = u_1r^{n-1}$.

Method 2

General solution is $a_n = c(-3)^n$

$a_1 = 6 \Rightarrow 6 = c(-3)^1 \Rightarrow c = -2$

Therefore, the particular solution is $a_n = -2(-3)^n$.

Write a general solution with an arbitrary constant, then use the initial condition to find the value of the constant.

Problem-solving

The two solutions are equivalent:

$$-2(-3)^n = -2(-3)(-3)^{n-1} = 6(-3)^{n-1}$$

You can find solutions to some non-homogeneous linear recurrence relations using back-substitution.

Example 9

Find a solution to the recurrence relation $u_n = u_{n-1} + n$, $n \geqslant 1$, with $u_0 = 0$.

Using iteration,

$u_n = u_{n-1} + n$

$= (u_{n-2} + (n - 1)) + n$

$\vdots$

$= u_0 + 1 + 2 + \ldots + n$

$= u_0 + \sum_{r=1}^{n} r$

$= \frac{n(n + 1)}{2}$

Therefore, the closed form for this recurrence relation is $u_n = \frac{n(n + 1)}{2}$

Replace n by $n - 1$ in $u_n = u_{n-1} + n$, then substitute.

$u_0 = 0$ and the sum of the first n integers is $\frac{n(n + 1)}{2}$ ← Core Pure Book 1, Section 3.1

You can apply this method to any recurrence relation of the form $u_n = u_{n-1} + g(n)$:

$u_n = u_{n-1} + g(n)$

$= (u_{n-2} + g(n - 1)) + g(n)$

$= ((u_{n-3} + g(n - 2)) + g(n - 1)) + g(n)$

$\vdots$

$= u_0 + \sum_{r=1}^{n} g(r)$

- **The solution to the first-order non-homogeneous linear recurrence relation $u_n = u_{n-1} + g(n)$ is given by $u_n = u_0 + \sum_{r=1}^{n} g(r)$.**

Watch out If u_1 is given instead of u_0, the solution would be $u_1 + \sum_{r=2}^{n} g(r)$

Example 10

Solve the following recurrence relations.

a $u_n = u_{n-1} + 2n + 1$, $n \geqslant 0$, with $u_0 = 7$

b $u_n = u_{n-1} + 5^n$, $n \in \mathbb{N}$, with $u_1 = 3$

a $u_n = u_0 + \sum_{r=1}^{n} g(r)$

$= 7 + \sum_{r=1}^{n} (2r + 1)$

$= 7 + 2\sum_{r=1}^{n} r + \sum_{r=1}^{n} 1$

$= 7 + n(n + 1) + n$

$= n^2 + 2n + 7$

Use the formula for the solution to a recurrence relation of the form $u_n = u_{n-1} + g(n)$.

Use the standard results for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} 1$. ← Core Pure Book 1, Section 3.1

b Method 1

$u_0 = u_1 - 5^1 = -2$

$u_n = u_0 + \sum_{r=1}^{n} g(r)$

$= -2 + \sum_{r=1}^{n} 5^r$

$= -2 + \frac{5(1 - 5^n)}{1 - 5}$

Method 2

$u_n = u_1 + \sum_{r=2}^{n} g(r) = 3 + \sum_{r=2}^{n} 5^r$

$= 3 + \frac{5^2(1 - 5^{n-1})}{1 - 5}$

$= -2 - \frac{5}{4} + \frac{5^{n+1}}{4}$

$= \frac{1}{4}(5^{n+1}) - \frac{13}{4}$

The initial condition is given in terms of u_1, so you need to find an expression for u_0 before you can use the formula.

$\sum_{r=1}^{n} 5^r$ is a geometric series with n terms, first term 5 and common ratio 5. ← Pure Year 2, Chapter 3

If you need to solve a recurrence relation of the form $u_n = au_{n-1} + g(n)$, where $a \neq 1$, back substitution gets more complicated. You can solve non-homogeneous recurrence relations of this form by first considering the general solution to the corresponding homogeneous recurrence relation, $u_n = au_{n-1}$. This general solution is called the **complementary function (C.F.)**. You then need to add a **particular solution (P.S.)** to the recurrence relation.

Links The particular solution plays a similar role to the particular integral which is used when solving a second-order linear differential equation. → Core Pure Book 2, Section 7.3

- **When solving a recurrence relation of the form $u_n = au_{n-1} + g(n)$, the form of the particular solution will depend on g(n):**

Form of g(n)	Form of particular solution
p with $a \neq 1$	λ
$pn + q$ with $a \neq 1$	$\lambda n + \mu$
kp^n with $p \neq a$	λk^n
ka^n	λna^n

Watch out This particular solution will satisfy the whole recurrence relation but will not necessarily satisfy the initial condition.

- **To solve the recurrence relation $u_n = au_{n-1} + g(n)$,**
  - **Find the complementary function (C.F.), which is the general solution to the associated homogeneous recurrence relation $u_n = au_{n-1}$.**
  - **Choose an appropriate form for a particular solution (P.S.) then substitute into the original recurrence relation to find the values of any coefficients.**
  - **The general solution is u_n = C.F. + P.S. = ca^n + P.S.**
  - **Use the initial condition to find the value of the arbitrary constant.**

Watch out The term **particular solution** is also sometimes used to refer to the final solution given the initial condition. Both versions satisfy the recurrence relation but only the final solution satisfies the initial condition.

You can use this method when $a = 1$, but in this case the complementary function is a constant, so you need to find a particular solution with no constant terms. You can do this by multiplying the particular solution by n:

Form of g(n)	Form of particular solution
p with $a = 1$	λn
$pn + q$ with $a = 1$	$\lambda n^2 + \mu n$

Note For recurrence relations of the form $u_n = u_{n-1} + p$ or $u_n = u_{n-1} + pn + q$, it is usually easier to use the summation formula given on page x.

Example 11

Solve the recurrence relation $u_n = 3u_{n-1} + 2n$, $n \in \mathbb{Z}^+$, with $u_1 = 3$.

Associated homogeneous recurrence relation is $u_n = 3u_{n-1}$

Complementary function: $u_n = c(3^n)$

Particular solution: $u_n = \lambda n + \mu\lambda$

$$u_n = 3u_{n-1} + 2n$$

$$\lambda n + \mu = 3(\lambda(n-1) + \mu) + 2n$$

$$\lambda n + \mu = 3\lambda n - 3\lambda + 3\mu + 2n$$

$$0 = (2\lambda + 2)n + (2\mu - 3\lambda)$$

$\Rightarrow 2\lambda + 2 = 0$ and $2\mu - 3\lambda = 0$

$\Rightarrow \lambda = -1, \mu = -\frac{3}{2}$

So a particular solution to the recurrence relation is $u_n = -n - \frac{3}{2}$

The general solution is $u_n = c(3^n) - n - \frac{3}{2}$

Since $u_1 = 3$,

$3 = c(3^1) - 1 - \frac{3}{2} \Rightarrow c = \frac{11}{6}$

The solution is $u_n = \frac{11}{6}(3^n) - n - \frac{3}{2}$

Find the general solution to the associated homogeneous recurrence relation. This is the **complementary function (C.F.)**.

g(n) is of the form $pn + q$, so look for a particular solution of the form $\lambda n + \mu$. You need to include the constant term even though g(n) does not have a constant term.

Substitute $u_n = \lambda n + \mu$ and $u_{n-1} = \lambda(n-1) + \mu$ into the full recurrence relation.

Problem-solving

Simplify, and group together coefficients of n and constant coefficients. Since the values of λ and μ must satisfy the recurrence relation for **any value of n**, you can consider it as an identity. This means that you can equate coefficients with the same power of n on both sides.

Solve these two equations simultaneously.

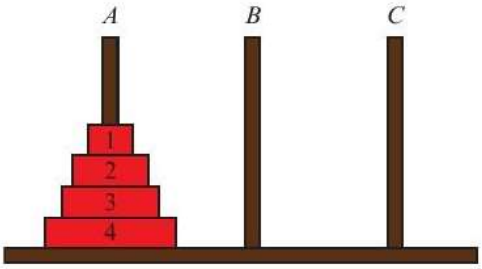
General solution = C.F. + P.S.

Use the initial condition $u_1 = 3$ to find the value of c.

You could prove this solution using mathematical induction. However, you don't need to do this in your exam unless you are explicitly asked to.

Example 12

The Tower of Hanoi puzzle involves transferring a pile of different sized disks from one peg to another using an intermediate peg.



The rules are as follows:

- Only one disk at a time can be moved.
- A disk can only be moved if it is the top disk on a pile.
- A larger disk can never be placed on a smaller one.

a Find the minimum number of moves needed to transfer two disks from one peg to another.

b Show that three disks can be transferred from one peg to another in 7 moves.

c Explain why the minimum number of moves, d_n, needed to transfer n disks from one peg to another satisfies the recurrence relation $d_n = 2d_{n-1} + 1$, with $d_1 = 1$.

d Solve this recurrence relation for d_n.

e Hence determine the minimum number of moves needed to transfer 15 disks from one peg to another.

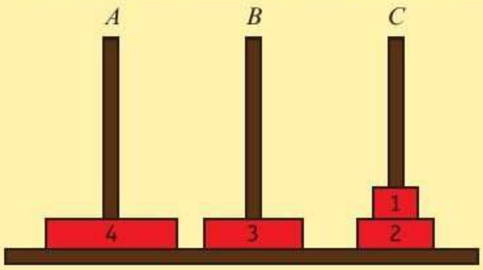
a 3 moves

For example, move disk 1 from A to B, disk 2 from A to C, then disk 1 from B to C.

b

Move number	Disk	From	To
1	1	A	B
2	2	A	C
3	1	B	C
4	3	A	B
5	1	C	A
6	2	C	B
7	1	A	B

After move 4 the disks are arranged as follows:



c Before you can move the largest disk (disk n), you must have transferred all the other disks to a single peg, say C. This requires d_{n-1} moves. You then move disk n in 1 move, to peg B. Finally, transfer the other disks to be on top of disk n, on peg B. This requires a further d_{n-1} moves. So the total number of moves is

$$d_n = d_{n-1} + 1 + d_{n-1} = 2d_{n-1} + 1$$

One disk can be transferred in one move so $d_1 = 1$.

Make sure you explain why the initial condition is true as well.

d Associated homogeneous recurrence relation: $d_n = 2d_{n-1}$
Complementary function: $d_n = c(2^n)$
Particular solution: $d_n = \lambda$
$d_n = 2d_{n-1} + 1$
$\lambda = 2\lambda + 1$
$\lambda = -1$
So a particular solution to the recurrence relation is $d_n = -1$.
The general solution is $d_n = c(2^n) - 1$
Since $d_1 = 1$,
$1 = c(2^1) - 1 \Rightarrow c = 1$
The solution is $d_n = 2^n - 1$

e $d_{15} = 2^{15} - 1 = 32767$

Find the general solution to the associated homogeneous recurrence relation.

The recurrence relation is of the form $u_n = au_{n-1} + g(n)$ with $a \neq 1$ and $g(n) = p$, a constant, so try a particular solution of the form $u_n = \lambda$.

Substitute $d_n = \lambda$ and $d_{n-1} = \lambda$ into the full recurrence relation and solve to find λ.

General solution = C.F. + P.S.

Use the initial condition $d_1 = 1$ to find the value of the arbitrary constant, c.

Exercise 4B

1 Find the solution to each of the following recurrence relations.

a $u_n = 2u_{n-1}$, with $u_0 = 5$

b $b_n = \frac{5}{2}b_{n-1}$, with $b_1 = 4$

c $d_n = -\frac{11}{10}d_{n-1}$, with $d_1 = 10$

d $x_{n+1} = -3x_n$, with $x_0 = 2$

2 Find a closed form for the sequences defined by the following recurrence relations.

a $u_n = u_{n-1} + 3$, with $u_0 = 5$

b $x_n = x_{n-1} + n$, with $x_0 = 2$

c $y_n = y_{n-1} + n^2 - 2$, with $y_0 = 3$

d $s_{n+1} = s_n + 2n - 1$, with $s_0 = 1$

Watch out In part **d**, the summation indices are slightly different, so this recurrence relation is not in the form $u_n = u_{n-1} + g(n)$. If you substitute n for $n - 1$ throughout the recurrence relation you can use the formula $u_n = u_0 + \sum_{r=1}^{n} g(r)$

3 Solve each of the following recurrence relations.

a $a_n = 2a_{n-1} + 1$, with $a_1 = 1$

b $u_n = -u_{n-1} + 2$, with $u_1 = 3$

c $h_n = 3h_{n-1} + 5$, with $h_0 = 1$

d $b_n = -2b_{n-1} + 6$, with $b_1 = 3$

Hint In each case, use a constant particular solution of the form λ.

(E) **4** In a league of n football teams, each team plays every other team exactly once. In total, g_n matches are played.

a Explain why $g_n = g_{n-1} + n - 1$, and write down a suitable initial condition for this recurrence relation. **(3 marks)**

b By solving your recurrence relation, show that $g_n = \frac{n(n-1)}{2}$ **(4 marks)**

5 **a** Find the general solution to the recurrence relation $u_n = 4u_{n-1} - 1$, $n \geqslant 2$

b Hence or otherwise find the particular solution given that:

i $u_1 = 3$ **ii** $u_1 = 0$ **iii** $u_1 = 200$

E/P **6** **a** Find the general solution to the recurrence relation $u_n = 3u_{n-1} + n$, $n > 1$. **(3 marks)**

b Given that $u_1 = 5$, find the particular solution to this recurrence relation. **(1 mark)**

E **7** A sequence is defined by the recurrence relation $u_{n+1} = 0.6u_n + 4$, with $u_0 = 7$.

a Find u_3. **(1 mark)**

b Find a closed form for the recurrence relation. **(3 marks)**

c Find the smallest value of n for which $u_n > 9.9$. **(1 mark)**

E/P **8** The deer population in a forest is estimated to drop by 5% each year.
Each year, 20 additional deer are introduced to the forest.
The initial deer population is 200, and the population after n years is given by D_n.

a Write down a recurrence relation for D_n. **(3 marks)**

b By solving your recurrence relation, find an expression for D_n in terms of n. **(3 marks)**

c Describe the behaviour of the deer population in the long term. **(1 mark)**

E **9** Solve the recurrence relation $u_n - 4u_{n-1} + 3 = 0$, with $u_0 = 7$. **(3 marks)**

E **10** A sequence of numbers satisfies the recurrence relation

$$u_n = u_{n-1} + 2^n,\ n \geqslant 2, \text{ with } u_1 = 5$$

Find a closed form for u_n. **(3 marks)**

E **11** Solve the recurrence relation $u_n = 4u_{n-1} + 2n$, with $u_0 = 7$. **(4 marks)**

E/P **12** A sequence satisfies the recurrence relation $u_n = 2u_{n-1} - 1005$, with $u_0 = 1000$.

a Solve the recurrence relation to find a closed form for u_n. **(4 marks)**

b Hence, or otherwise, find the first negative term in the sequence. **(3 marks)**

E/P **13** **a** Find the general solution to the recurrence relation $u_n = 2u_{n-1} - 2^n$, $n \geqslant 2$. **(4 marks)**

b Find the particular solution to this recurrence relation given that $u_1 = 3$. **(1 mark)**

E/P **14** A sequence is defined by the recurrence relation $u_n = ku_{n-1} + 1$, $k \neq 1$, with $u_0 = 0$.

a Find the value of u_1, u_2, and u_3 in terms of k. **(2 marks)**

b Find a closed form for this sequence. **(3 marks)**

c Describe the behaviour of the sequence as n gets very large in the cases when:

i $k > 1$ **ii** $-1 < k < 1$ **iii** $k = -1$ **iv** $k < -1$ **(4 marks)**

E/P **15** A sequence is defined by the recurrence relation

$$a_n = a_{n-1} + 6n + 1,\ n \in \mathbb{Z}^+, \text{ with } a_0 = 2$$

a Find $\sum_{r=1}^{n}(6r + 1)$ **(2 marks)**

b Hence, or otherwise, find a closed form for this sequence. **(3 marks)**

c Given that $a_n = 561$, find the value of n. **(2 marks)**

E/P **16** **a** Solve the recurrence relation $u_n = u_{n-1} - 6n^2$, with $u_0 = 89$. **(3 marks)**

b Hence, or otherwise, find the first negative term of the sequence. **(2 marks)**

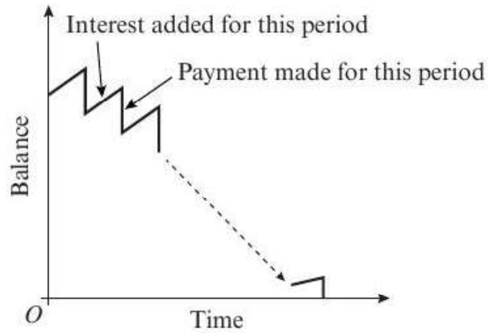
c Explain why every term of the sequence is an odd number. **(2 marks)**

E/P **17** **a** Solve the recurrence relation $u_n = u_{n-1} - 2n$, with $u_0 = 3$. **(2 marks)**

b Show that -103 is not a term of the sequence. **(2 marks)**

c Given that $u_k = -459$, find the value of k. **(2 marks)**

E/P **18** Alison borrows £2000 on her credit card. She intends to pay it back by making 18 monthly payments. At the end of each month, interest of 1.5% is added to the loan balance, and Alison's monthly payment of £P is deducted from the loan balance. The graph illustrates how the balance of the loan will change over time.



a Write a recurrence relation for the balance of the loan at the end of n months. **(3 marks)**

b Find a solution to your recurrence relation, giving your answer in terms of P. **(3 marks)**

Alison wants the balance of the loan to be zero after she makes her 18th payment.

c Find the value of P that will make this the case. **(3 marks)**

Challenge

A **restricted** Tower of Hanoi problem requires a player to move a pile of disks of different sizes from peg A to peg C. The rules are as follows:

- Only one disk at a time can be moved.
- A disk can only be moved if it is the top disk on a pile.
- A larger disk can never be placed on a smaller one.
- **Disks can only be moved a distance of one peg at a time.**

Let H_n be the minimum number of moves needed to transfer n discs from peg A to peg C.

a Explain why $H_1 = 2$.

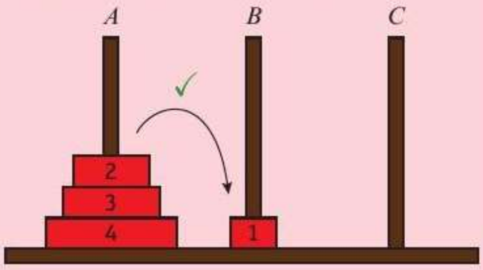
b Show that 2 disks can be moved from peg A to peg C in 8 moves.

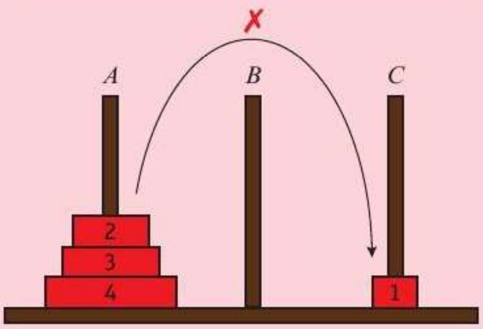
c Explain why H_n satisfies a recurrence relation of the form $H_n = aH_{n-1} + b$, and determine the values of a and b.

d **i** Solve this recurrence relation for H_n.

ii Hence determine the minimum number of moves needed to transfer 10 disks from peg A to peg C.

Watch out The fourth rule means that moves between pegs A and B, and pegs B and C are allowed, but direct moves between pegs A and C are not.





4.3 Solving second-order recurrence relations

A

In this section you will learn how to solve second-order linear recurrence relations.

- **A second-order linear recurrence relation can be written in the form $u_n = au_{n-1} + bu_{n-2} + g(n)$, where a and b are real constants.**
  - **If $g(n) = 0$, then the equation is homogeneous.**

Example 13

Consider the recurrence relation $u_n = 2u_{n-1} - u_{n-2}$, $n \geqslant 2$.

Verify that the following particular solutions satisfy this recurrence relation.

a $u_n = 3n$ **b** $u_n = 5$ **c** $u_n = 3n + 5$

a $u_n = 3n$, $u_{n-1} = 3(n-1) = 3n - 3$
$u_{n-2} = 3(n-2) = 3n - 6$
$2u_{n-1} - u_{n-2} = 2(3n-3) - (3n-6) = 3n = u_n$
So $u_n = 3n$ satisfies the recurrence relation.

Find expressions for u_{n-1} and u_{n-2} and substitute them into the RHS of the recurrence relation.

b $u_n = 5$, $u_{n-1} = 5$, $u_{n-2} = 5$
$2u_{n-1} - u_{n-2} = 2 \times 5 - 5 = 5 = u_n$
So $u_n = 5$ satisfies the recurrence relation.

c $u_n = 3n + 5$, $u_{n-1} = 3(n-1) + 5 = 3n + 2$
$u_{n-2} = 3(n-2) + 5 = 3n - 1$
$2u_{n-1} - u_{n-2} = 2(3n+2) - (3n-1) = 3n + 5 = u_n$
So $u_n = 3n + 5$ satisfies the recurrence relation.

- **If $u_n = F(n)$ and $u_n = G(n)$ are particular solutions to a linear recurrence relation, then $u_n = aF(n) + bG(n)$, where a and b are constants, is also a solution.**

You can solve a **second-order homogeneous linear** recurrence relation by looking for solutions of the form $u_n = Ar^n$, where A is an arbitrary non-zero constant.

Suppose that $u_n = Ar^n$ is a solution to the recurrence relation $u_n = au_{n-1} + bu_{n-2}$.

Then $Ar^n = Aar^{n-1} + Abr^{n-2}$
$\Rightarrow r^2 - ar - b = 0$

Multiply both sides by r^{2-n} and simplify.

This quadratic equation is called the **auxiliary equation** of the recurrence relation. $u_n = Ar^n$ is a solution to the recurrence relation if and only if r is a root of this equation.

Notation The auxiliary equation is sometimes called the **characteristic equation**.

However, a second-order recurrence relation requires **two initial conditions** to fully define the sequence. As such, the general solution to a second-order recurrence relation requires **two arbitrary constants**. You can formulate a general solution with two arbitrary constants by adding multiples of two different solutions.

A

- **You can find a general solution to a second-order homogeneous linear recurrence relation, $u_n = au_{n-1} + bu_{n-2}$, by considering the auxiliary equation**

  $r^2 - ar - b = 0$

  You need to consider three different cases:

  - **Case 1: Distinct real roots**
    If the auxiliary equation has distinct real roots α and β, then the general solution will have the form $u_n = A\alpha^n + B\beta^n$ where A and B are arbitrary constants.

  - **Case 2: Repeated root**
    If the auxiliary equation has a repeated real root α, then the general solution will have the form $u_n = (A + Bn)\alpha^n$ where A and B are arbitrary constants.

  - **Case 3: Complex roots**
    If the auxiliary equation has two complex roots $\alpha = re^{i\theta}$ and $\beta = re^{-i\theta}$, then the general solution will have the form $u_n = r^n(A\cos n\theta + B\sin n\theta)$, or $u_n = A\alpha^n + B\beta^n$, where A and B are arbitrary constants.

Links These three cases are similar to the cases you consider when solving a differential equation of the form $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$.

→ Core Pure Book 2, Section 7.2

Example 14

a Find a general solution to the recurrence relation

$u_n = 2u_{n-1} + 8u_{n-2},\ n \geqslant 2$

b Verify that your general solution from part **a** satisfies the recurrence relation.

c Given than $u_0 = 4$ and $u_1 = 10$, find a particular solution.

a $r^2 - 2r - 8 = 0$
$(r - 4)(r + 2) = 0$
$\Rightarrow r = 4$ or $r = -2$

Write down the auxiliary equation and solve it.

General solution is $u_n = A(4^n) + B(-2)^n$

The auxiliary equation has two distinct real roots, so the general solution has the form $u_n = A\alpha^n + B\beta^n$.

b $2u_{n-1} + 8u_{n-2}$
$= 2(A(4^{n-1}) + B(-2)^{n-1}) + 8(A(4^{n-2}) + B(-2)^{n-2})$
$= 2A(4^{n-1}) + 2B(-2)^{n-1} + 8A(4^{n-2}) + 8B(-2)^{n-2}$
$= 2A(4^{n-1}) + 2B(-2)^{n-1} + 2A(4^{n-1}) - 4B(-2)^{n-1}$
$= 4A(4^{n-1}) - 2B(-2)^{n-1}$
$= A(4^n) + B(-2)^n$
$= u_n$

Substitute your general solution into the RHS of the recurrence relation, then take out factors of 4 and −2 to write the expression in terms of multiples of 4^n and $(-2)^n$.

c $u_0 = 4 \Rightarrow A(4^0) + B(-2)^0 = 4 \Rightarrow A + B = 4$
$u_1 = 10 \Rightarrow A(4^1) + B(-2)^1 = 10 \Rightarrow 4A - 2B = 10$
Solving simultaneously, $A = 3$ and $B = 1$
So the solution is $u_n = 3(4^n) + (-2)^n$

Use the initial conditions to write two simultaneous equations in A and B. Solve these to find the values of the arbitrary constants.

Example 15

A Solve the recurrence relation $a_n = 3a_{n-1} - 2a_{n-2}$, with $a_1 = 5$ and $a_2 = 3$.

$r^2 - 3r + 2 = 0$
$(r-1)(r-2) = 0$
$\Rightarrow r = 1$ or $r = 2$

Write down the auxiliary equation and solve it.

So the general solution is $a_n = A(1^n) + B(2^n)$
$= A + B(2^n)$

One of the roots is 1, so one of the terms in the general solution will be constant.

$\left.\begin{array}{l} a_1 = A + 2B = 5 \\ a_2 = A + 4B = 3 \end{array}\right\} \Rightarrow A = 7,\ B = -1$

Use the initial conditions to find the arbitrary constants.

So the solution is $a_n = 7 - 2^n$.

Problem-solving

You can check your answer by generating the first few terms of the sequence using the solution and the original recurrence relation. The first 5 terms here are 5, 3, −1, −9, and −25.

Example 16

Solve the recurrence relation $u_n = 4u_{n-1} - 4u_{n-2}$, with $u_0 = 1$ and $u_1 = 1$.

$r^2 - 4r + 4 = 0$
$(r-2)^2 = 0$
$\Rightarrow r = 2$

So the general solution is $u_n = (A + Bn)2^n$

The auxiliary equation has one repeated root, so the general solution is of the form $u_n = (A + Bn)\alpha^n$.

$u_0 = 1 \Rightarrow A = 1$
$u_1 = 1 \Rightarrow 2(A + B) = 1$
$2 + 2B = 1$
$B = -\frac{1}{2}$

Use the initial conditions to form two equations and solve these to find A and B.

The solution is $u_n = (1 - \frac{1}{2}n)2^n$.

You could also write this as $u_n = 2^n - n2^{n-1}$.

Example 17

a Find the general solution to the recurrence relation $u_n = 2u_{n-1} - 2u_{n-2}$.

b Given that $u_0 = 1$ and $u_2 = 2$, find the particular solution to the recurrence relation.

a $r^2 - 2r + 2 = 0$

$\Rightarrow r = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm \mathrm{i} = \sqrt{2}e^{\pm\frac{\pi}{4}}$

The auxiliary equation has distinct complex roots. These must be a conjugate pair, so you can write them in the form $x \pm \mathrm{i}y$ or $re^{\pm \mathrm{i}\theta}$.

← Core Pure Book 2, Chapter 1

Form 1: $u_n = A(1 + \mathrm{i})^n + B(1 - \mathrm{i})^n$

Form 2: $u_n = (\sqrt{2})^n\left(C\cos\dfrac{n\pi}{4} + D\sin\dfrac{n\pi}{4}\right)$

The form $u_n = r^n(A\cos n\theta + B\sin n\theta)$ only uses real numbers.

A

b Using form 1:

$u_0 = A(1 + i)^0 + B(1 - i)^0 = 1$

$\Rightarrow A + B = 1$ (1)

$u_1 = A(1 + i)^1 + B(1 - i)^1 = 2$

$\Rightarrow A + B + (A - B)i = 2$

$\Rightarrow (A - B)i = 1$

$\Rightarrow A - B = -i$ (2)

Solving (1) and (2):

$A = \frac{1 - i}{2}$ and $B = \frac{1 + i}{2}$

So the particular solution is

$u_n = \left(\frac{1 - i}{2}\right)(1 + i)^n + \left(\frac{1 + i}{2}\right)(1 - i)^n$

Using form 2:

$u_0 = C(\sqrt{2})^0 \cos 0 + D(\sqrt{2})^0 \sin 0 = 1$

$\Rightarrow C = 1$

$u_1 = C(\sqrt{2})^1 \cos\frac{\pi}{4} + C(\sqrt{2})^1 \sin\frac{\pi}{4} = 2$

$\Rightarrow 1 \times \sqrt{2} \times \frac{\sqrt{2}}{2} + D \times \sqrt{2} \times \frac{\sqrt{2}}{2} = 2$

$\Rightarrow D = 1$

So the particular solution is

$u_n = (\sqrt{2})^n\left(\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right)$

Watch out The values of the arbitrary constants will be different depending on which form you use.

Use the initial conditions to find the values of the arbitrary constants A and B. If you are using this form of the general solution, the arbitrary constants can be **complex**.

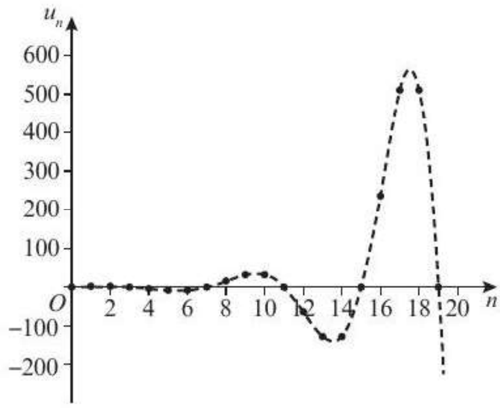
Problem-solving

You can simplify this to $u_n = (1 + i)^{n-1} + (1 - i)^{n-1}$ by writing, for example,

$\left(\frac{1 - i}{2}\right)(1 + i)^n = \frac{(1 - i)(1 + i)}{2}(1 + i)^{n-1} = (1 + i)^{n-1}$

With this form of the general solution, both arbitrary constants will be real numbers.

You can use the addition formula for sine to write the solution to the recurrence relation in Example 17 as $u_n = (\sqrt{2})^{n+1}\sin\frac{(n + 1)\pi}{4}$. This helps you to see that the sequence oscillates between positive and negative values, with the magnitude of the oscillations increasing as n increases. The graph shows the sequence from u_0 to u_{19}. Note that the terms only exist for integer values of n and that, in this case, u_n is always an integer.



Example 18

The Fibonacci sequence is defined recursively as

$$F_n = F_{n-1} + F_{n-2},\ n > 2, \text{ with } F_1 = 1 \text{ and } F_2 = 1$$

Find a closed form for F_n.

$r^2 - r - 1 = 0$

$\Rightarrow r = \frac{1 + \sqrt{5}}{2}$ or $r = \frac{1 - \sqrt{5}}{2}$

So the general solution is

$F_n = A\left(\frac{1 + \sqrt{5}}{2}\right)^n + B\left(\frac{1 - \sqrt{5}}{2}\right)^n$

Solve the auxiliary equation.

Using the initial conditions,

$F_1 = 1 \Rightarrow A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$

$A(1+\sqrt{5}) + B(1-\sqrt{5}) = 2$ (1)

$F_2 = 1 \Rightarrow A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$

$A(3+\sqrt{5}) + B(3-\sqrt{5}) = 2$ (2)

Solving (1) and (2) simultaneously,

$A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$

The solution is

$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$

Watch out You can solve these simultaneous equations quickly using your calculator. However, make sure you show enough working to demonstrate that you have used the initial conditions to generate two simultaneous equations.

You can solve **non-homogeneous** linear second-order recurrence relations by considering the complementary function (C.F.) and finding a suitable particular solution (P.S.).

- **To solve the recurrence relation $u_n = au_{n-1} + bu_{n-2} + g(n)$,**
  - **Find the complementary function, which is the general solution to the associated homogeneous recurrence relation $u_n = au_{n-1} + bu_{n-2}$.**
  - **Choose an appropriate form for a particular solution then substitute into the original recurrence relation to find the values of any coefficients.**
  - **The general solution is u_n = C.F. + P.S.**
  - **Use the initial conditions to find the values of the arbitrary constants.**

The form of the particular solution will depend on $g(n)$.

- **For the recurrence relation $u_n = au_{n-1} + bu_{n-2} + g(n)$, with auxiliary equation with roots α and β, try the following forms for a particular solution:**

Form of $g(n)$	Form of particular solution
p with $\alpha, \beta \neq 1$	λ
$pn + q$, with $\alpha, \beta \neq 1$	$\lambda n + \mu$
kp^n with $p \neq \alpha, \beta$	λp^n
p with $\alpha = 1, \beta \neq 1$	λn
$pn + q$ with $\alpha = 1, \beta \neq 1$	$\lambda n^2 + \mu n$
p with $\alpha = \beta = 1$	λn^2
$pn + q$ with $\alpha = \beta = 1$	$\lambda n^3 + \mu n^2$
$k\alpha^n$ with $\alpha \neq \beta$	$\lambda n \alpha^n$
$k\alpha^n$ with $\alpha = \beta$	$\lambda n^2 \alpha^n$

Watch out The particular solution cannot have any terms in common with the complementary function of the associated homogeneous recurrence relation. The last six lines of this table, shown shaded, are special cases to avoid this.
When $\alpha = 1$, multiply the expected form of the particular solution by n.
When $\alpha = 1$ and $\beta = 1$, multiply the expected form of the particular solution by n^2.

Example 19

Solve the recurrence relation

$$a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n,\ n > 0, \text{ with } a_0 = 2 \text{ and } a_1 = -1$$

A

Associated homogeneous recurrence relation:

$a_{n+2} + 4a_{n+1} + 3a_n = 0$

$r^2 + 4r + 3 = 0$

$(r + 1)(r + 3) = 0$

$\Rightarrow r = -1$ or $r = -3$

So the complementary function is

$a_n = A(-1)^n + B(-3)^n$

Try particular solution $a_n = \lambda(-2)^n$:

$$a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n$$

$$\lambda(-2)^{n+2} + 4\lambda(-2)^{n+1} + 3\lambda(-2)^n = 5(-2)^n$$

$$4\lambda - 8\lambda + 3\lambda = 5$$

$$\lambda = -5$$

So a particular solution is $a_n = -5(-2)^n$, and the general solution to the recurrence relation is

$a_n = A(-1)^n + B(-3)^n - 5(-2)^n$

$a_0 = A(-1)^0 + B(-3)^0 - 5(-2)^0 \Rightarrow A + B - 5 = 2$

$$A + B = 7$$

$a_1 = A(-1)^1 + B(-3)^1 - 5(-2)^1 = -1$

$$-A - 3B + 10 = -1$$

$$A + 3B = 11$$

$$\left.\begin{aligned} A + B &= 7 \\ A + 3B &= 11 \end{aligned}\right\} \Rightarrow A = 5 \text{ and } B = 2$$

So the solution is $a_n = 5(-1)^n + 2(-3)^n - 5(-2)^n$

Find the general solution to the associated homogeneous recurrence relation.

Divide both sides of the equation by $(-2)^n$ and simplify.

General solution = C.F. + P.S.

The values of A and B can be found by using the initial conditions.

Problem-solving

Check your answer using $n = 0, 1, 2$.

Closed form:

$a_0 = 5(-1)^0 + 2(-3)^0 - 5(-2)^0 = 2$

$a_1 = 5(-1)^1 + 2(-3)^1 - 5(-2)^1 = -1$

$a_2 = 5(-1)^2 + 2(-3)^2 - 5(-2)^2 = 3$

Recurrence relation:

$a_0 = 2, a_1 = -1$

$a_2 + 4a_1 + 3a_0 = 5(-2)^0 \Rightarrow a_2 = 5 + 4 - 6 = 3.$

Example 20

Find the general solution to $s_n = 3s_{n-1} + 4s_{n-2} + 4^n$.

Associated homogeneous recurrence relation:

$s_n - 3s_{n-1} - 4s_{n-2} = 0$

$r^2 - 3r - 4 = 0$

$(r + 1)(r - 4) = 0$

$\Rightarrow r = -1$ or $r = 4$

So the complementary function is

$s_n = A(-1)^n + B(4^n)$

Try particular solution $s_n = \lambda n(4^n)$:

$$s_n = 3s_{n-1} + 4s_{n-2} + 4^n$$

$$\lambda n(4^n) = 3\lambda(n-1)(4^{n-1}) + 4\lambda(n-2)(4^{n-2}) + 4^n$$

$$\lambda n(4^n) = \tfrac{3}{4}\lambda(n-1)(4^n) + \tfrac{1}{4}\lambda(n-2)(4^n) + 4^n$$

$$\lambda n = \tfrac{3}{4}\lambda(n-1) + \tfrac{1}{4}\lambda(n-2) + 1$$

$$\lambda n = \tfrac{3}{4}\lambda n - \tfrac{3}{4}\lambda + \tfrac{1}{4}\lambda n - \tfrac{1}{2}\lambda + 1$$

So $-\frac{5}{4}\lambda + 1 = 0 \Rightarrow \lambda = \frac{4}{5}$

So a particular solution is $\frac{4}{5}n(4^n)$, and the general solution is

$s_n = A(-1)^n + B(4^n) + \frac{4}{5}n(4^n)$

Write down the auxiliary equation and solve it.

Watch out You cannot use a particular solution of the form $\lambda(4^n)$ because the complementary function already features a 4^n term. Look for a particular solution of the form $\lambda n(4^n)$ instead.

Substitute the particular solution into the full recurrence relation and solve to find λ.

This question only asks for the general solution, so leave your answer in this form with two arbitrary constants.

Exercise 4C

A

1 Consider the recurrence relation $u_n = 5u_{n-1} + 6u_{n-2}$. Verify that each of the following solutions satisfies this recurrence relation.

a $u_n = (-1)^n$ **b** $u_n = 6^n$

c $u_n = A(-1)^n + B(6^n)$, where A and B are arbitrary constants.

2 Consider the recurrence relation $u_n - 6u_{n-1} + 9u_{n-2} = 0$. Verify that each of the following solutions satisfies this recurrence relation.

a $u_n = 5(3^n)$ **b** $u_n = -n3^n$ **c** $u_n = 5(3^n) - n3^n$

3 Consider the recurrence relation $u_{n+2} + u_n = 0$. Verify that each of the following solutions satisfies this recurrence relation.

a $u_n = \cos\left(n\frac{\pi}{2}\right)$ **b** $u_n = \sin\left(n\frac{\pi}{2}\right)$

c $u_n = A\cos\left(n\frac{\pi}{2}\right) + B\sin\left(n\frac{\pi}{2}\right)$, where A and B are arbitrary constants.

(P) 4 $u_n = \text{F}(n)$ and $u_n = \text{G}(n)$ are particular solutions to the linear homogeneous recurrence relation

$$u_n = au_{n-1} + bu_{n-2}$$

Show that $u_n = c\text{F}(n) + d\text{G}(n)$ is also a solution, where c and d are arbitrary constants.

5 Find the general solution to each of the following recurrence relations.

a $a_n = 2a_{n-1} - a_{n-2}$ **b** $u_n - 3u_{n-1} + 2u_{n-2} = 0$

c $x_n = 6x_{n-1} - 9x_{n-2}$ **d** $t_n = 4t_{n-1} - 5t_{n-2}$

Hint Your general solutions will each contain two arbitrary constants.

(P) 6 The recurrence relation $u_{n+2} + au_{n+1} + bu_n = 0$, where a and b are real constants, has general solution $u_n = D + E(7^n)$, where D and E are arbitrary constants.
Find the values of a and b.

7 Solve each of the following recurrence relations.

a $a_n = 5a_{n-1} - 6a_{n-2}$, with $a_0 = 2$ and $a_1 = 5$

b $u_n = 6u_{n-1} - 9u_{n-2}$, $n \geqslant 3$, with $u_1 = 2$ and $u_2 = 5$

c $s_n = 7s_{n-1} - 10s_{n-2}$, $n \geqslant 2$, with $s_0 = 4$ and $s_1 = 17$

d $u_n = 2u_{n-1} - 5u_{n-2}$, with $u_0 = 1$ and $u_1 = 5$

(E/P) 8 A sequence satisfies the recurrence relation $u_n = 5u_{n-1} - 4u_{n-2}$, with $u_0 = 20$ and $u_1 = 19$.

a Solve the recurrence relation to find a closed form for u_n. **(5 marks)**

b Show that the sequence is decreasing, and that $u_n < 0$ for all $n \geqslant 3$. **(3 marks)**

(E/P) 9 **a** Find a closed form for the sequence defined by the recurrence relation $u_n = \sqrt{2}u_{n-1} - u_{n-2}$, with $u_0 = u_1 = 1$ **(5 marks)**

b Hence show that the sequence is periodic and state its period. **(3 marks)**

(E) 10 The nth Lucas number L_n, is defined by $L_n = L_{n-1} + L_{n-2}$, $n \geqslant 3$, with $L_1 = 1$, $L_2 = 3$.

a List the first 7 terms of the sequence. **(1 mark)**

b Show that a closed form for the nth Lucas number is $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$ **(5 marks)**

A **11** Find the general solution to each of the following recurrence relations.

a $x_n = 5x_{n-1} - 6x_{n-2} + 1$ **b** $u_n - u_{n-1} - 2u_{n-2} = 2n$

c $a_{n+2} + 4a_{n+1} + 3a_n = 5(-2)^n$ **d** $a_{n+2} + 4a_{n+1} + 3a_n = 12(-3)^n$

e $a_{n+2} - 6a_{n+1} + 9a_n = 3^n$ **f** $u_n = 7u_{n-1} - 10u_{n-2} + 6 + 8n$

Problem-solving

Each of these recurrence relations will require a particular solution. Look at the table on page x to determine the correct form for the particular solution.

12 Solve each of the following recurrence relations.

a $u_n = 2u_{n-1} + 3u_{n-2} + 1$, $n \geqslant 3$, with $u_1 = 3$ and $u_2 = 7$

b $a_{n+1} - 3a_n + 2a_{n-1} = 6(-1)^n$, with $a_0 = a_1 = 12$

c $u_n = 3u_{n-1} + 10u_{n-2} + 7 \times 5^n$, $n \geqslant 2$, with $u_0 = 4$ and $u_1 = 3$

d $x_n = 10x_{n-1} - 25x_{n-2} + 8 \times 5^n$, $n \geqslant 2$, with $x_0 = 6$ and $x_1 = 10$

E/P **13** Consider the recurrence relation $b_{n+2} + 4b_{n+1} + 4b_n = 7$.

a Find a constant k such that $b_n = k$ is a particular solution to this recurrence relation. **(2 marks)**

b Hence or otherwise, solve the recurrence relation given that $b_0 = 1$ and $b_1 = 2$. **(5 marks)**

E/P **14** **a** Find the general solution to the recurrence relation $u_n = 7u_{n-1} - 6u_{n-2} + 75$. **(4 marks)**

b Given that $u_0 = u_1 = 2$, find the particular solution. **(3 marks)**

E/P **15** Consider the recurrence relation $u_{n+2} - 6u_{n+1} + 9u_n = 7(3^n)$.

a Find a value of k such that $u_n = kn^2(3^n)$ is a particular solution to this recurrence relation. **(2 marks)**

b Find the general solution to $u_{n+2} - 6u_{n+1} + 9u_n = 0$. **(3 marks)**

c Hence, find the solution to $u_{n+2} - 6u_{n+1} + 9u_n = 7(3^n)$ given that $u_0 = 1$ and $u_1 = 4$. **(3 marks)**

E/P **16** A sequence of numbers satisfies the recurrence relation $u_n = u_{n-1} - u_{n-2}$, $n \geqslant 2$.

a Given that $u_0 = 0$ and $u_1 = 3$, show the solution to this recurrence relation can be written in the form $u_n = p\sin qn$, where p and q are exact real constants to be determined. **(6 marks)**

b Hence explain why the sequence u_n is periodic, and state its period. **(2 marks)**

E/P **17** A monkey sits at a typewriter and types strings of random letters. Unfortunately, the typewriter is broken, so the only keys that work are the letters A, B and C.

a Find the number of different strings of length 3 which do not contain consecutive letter As. **(2 marks)**

The number of different strings of length n which do not contain consecutive letter As is given by s_n.

b Find a recurrence relation for s_n in terms of s_{n-1} and s_{n-2}. **(3 marks)**

c **i** Solve your recurrence relation.

ii Find the number of strings of length 20 which do not contain consecutive letter As. **(3 marks)**

Challenge

1 Solve the recurrence relation $u_n = \sqrt{\dfrac{u_{n-2}}{u_{n-1}}}$, $u_0 = 8$, with $u_1 = \dfrac{1}{2\sqrt{2}}$

2 The sequence u_n satisfies the recurrence relation $u_n = au_{n-1} + bu_{n-2}$, with $u_0 = 0$ and $u_1 = k$, where a, b and k are real constants, and $k \neq 0$. Find values of a and b such that the sequence is periodic with period 12, and state the maximum and minimum values in the sequence in terms of k.

Hint Take logs of both sides, and then use a suitable substitution to form a linear recurrence relation.

4.4 Proving closed forms

You can prove that a closed form satisfies a given recurrence relation using mathematical induction.

Example 21

The minimum number of moves, d_n, needed to transfer n disks from one peg to another in the Tower of Hanoi problem is given by the recurrence relation $d_n = 2d_{n-1} + 1$, with $d_1 = 1$.

Prove, by induction, that $d_n = 2^n - 1$.

Basis step:

When $n = 1$, $d_1 = 2^1 - 1 = 1$

So the closed form is true for $n = 1$.

Assumption step:

Assume the closed form is true for $n = k$.

So $d_k = 2^k - 1$

Inductive step:

Using the recurrence relation,

$d_{k+1} = 2d_k + 1 = 2(2^k - 1) + 1$

$= 2 \times 2^k - 2 \times 1 + 1 = 2^{k+1} - 1$

So true for $n = k \Rightarrow$ true for $n = k + 1$, and true for $n = 1$. Therefore, by induction, the closed form $d_n = 2^n - 1$ is true for all $n \in \mathbb{N}$.

You are given that $d_1 = 1$. Use this to prove the basis step.

You need to assume that the closed form is true for $n = k$, then use the recurrence relation to show that it is true for $n = k + 1$.

← Core Pure Book 1, Chapter 8

Keep in mind what you are aiming to show. In this case, you need to show that $d_{k+1} = 2^{k+1} - 1$.

Replace d_k with its assumed value of $2^k - 1$.

Remember to write a conclusion, and state that you have used induction.

Example 22

A sequence u_n satisfies the recurrence relation $u_n = u_{n-1} + n$, with $u_0 = 0$.

Prove by induction that $u_n = \frac{n(n + 1)}{2}$, $n \geqslant 0$.

Basis step:

When $n = 0$, $u_1 = \frac{0(0 + 1)}{2} = 0$

So the closed form is true for $n = 0$.

Assumption step:

Assume the closed form is true for $n = k$:

$u_k = \frac{k(k + 1)}{2}$

Inductive step:

Using the recurrence relation $u_k = u_{k-1} + k$,

$u_{k+1} = u_k + k + 1 = \frac{k(k + 1)}{2} + k + 1$

$= \frac{k(k + 1) + 2(k + 1)}{2} = \frac{(k + 1)(k + 2)}{2}$

So true for $n = k \Rightarrow$ true for $n = k + 1$, and true for $n = 1$. Therefore, by induction, the closed form $u_n = \frac{n(n + 1)}{2}$ is true for all $n \in \mathbb{N}$.

$u_0 = 0$ is given, so you can begin your induction at $n = 0$.

Replace u_k by its assumed value of $\frac{k(k + 1)}{2}$

Keep in mind what you are aiming to show. In this case you need to show that $u_{k+1} = \frac{(k + 1)(k + 2)}{2}$

A You can adapt the technique of proof by mathematical induction to prove closed forms for second-order recurrence relations.

- **When you are proving the closed form of a second-order recurrence relation by mathematical induction, you need to:**
  - **show that the closed form is true for two consecutive values of n (basis step)**
  - **assume that the closed form is true for $n = k$ and $n = k - 1$ (assumption step), then show that it is true for $n = k + 1$ (inductive step).**

Example 23

A sequence a_n satisfies the recurrence relation $a_n = 2a_{n-1} + 8a_{n-2}$, with $a_0 = 4$, $a_1 = 10$.
Prove by induction that $a_n = 3(4^n) + (-2)^n$ for all non-negative integers n.

Basis step:
When $n = 0$, $a_0 = 3 \times 4^0 + (-2)^0 = 3 \times 1 + 1 = 4$
When $n = 1$, $a_1 = 3 \times 4^1 + (-2)^1 = 12 - 2 = 10$
So the closed form is true for $n = 0$ and $n = 1$.

Show that the closed form is true for **two consecutive** values of n. You are given $a_0 = 4$ and $a_1 = 10$, so substitute $n = 0$ and $n = 1$ into the closed form.

Assumption step:
Assume the closed form is true for $n = k$ and $n = k - 1$:
$a_k = 3(4^k) + (-2)^k$ and $a_{k-1} = 3(4^{k-1}) + (-2)^{k-1}$

Inductive step:
Using the recurrence relation $a_k = 2a_{k-1} + 8a_{k-2}$

$$
\begin{aligned}
a_{k+1} &= 2a_k + 8a_{k-1} \\
&= 2(3(4^k) + (-2)^k) + 8(3(4^{k-1}) + (-2)^{k-1}) \\
&= 2(3(4^k) + (-2)^k) + 8(3(4^{k-1}) + (-2)^{k-1}) \\
&= 6(4^k) + 2(-2)^k + 24(4^{k-1}) + 8(-2)^{k-1} \\
&= 6(4^k) - (-2)^{k+1} + 6(4^k) + 2(-2)^{k+1} \\
&= 12(4^k) + (-2)^{k+1} = 3(4^{k+1}) + (-2)^{k+1}
\end{aligned}
$$

So true for $n = k$ and $n = k - 1 \Rightarrow$ true for $n = k + 1$, and true for $n = 0$ and $n = 1$.
Therefore, by induction, the closed form for the nth term, $a_n = 3(4^n) + (-2)^n$, is true for all non-negative integers.

Keep in mind what you are aiming to show. In this case you need to show that $a_{k+1} = 3(4^{k+1}) + (-2)^{k+1}$

Replace a_k with its assumed value of $3(4^k) + (-2)^k$ and a_{k-1} with its assumed value of $3(4^{k-1}) + (-2)^{k-1}$

Simplify and factorise. Simplify by replacing $+2(-2)^k$ with $-(-2)(-2)^k = -(-2)^{k+1}$, $24(4^{k-1})$ with $6(4^k)$ and $8(-2)^{k-1}$ with $2(-2)^2(-2)^{k-1} = 2(-2)^{k+1}$

Example 24

The Fibonacci sequence is defined recursively by

$$u_n = u_{n-1} + u_{n-2},\ n > 2, \text{ with } u_1 = u_2 = 1$$

Show that the closed form for the nth term of the Fibonacci sequence is given by

$$u_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n\sqrt{5}},\ n > 2$$

A

Basis step:

When $n = 1$, $u_1 = \dfrac{(1+\sqrt{5})^1 - (1-\sqrt{5})^1}{2\sqrt{5}} = \dfrac{2\sqrt{5}}{2\sqrt{5}} = 1$

When $n = 2$, $u_2 = \dfrac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{2^2\sqrt{5}}$

$= \dfrac{1 + 2\sqrt{5} + 5 - 1 + 2\sqrt{5} - 5}{4\sqrt{5}} = \dfrac{4\sqrt{5}}{4\sqrt{5}} = 1$

So the closed form is true for $n = 1$ and $n = 2$.

Assumption step:

Assume the closed form is true for $n = k - 1$ and $n = k$.

So $u_k = \dfrac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}}$ and $u_{k-1} = \dfrac{(1+\sqrt{5})^{k-1} - (1-\sqrt{5})^{k-1}}{2^{k-1}\sqrt{5}}$

Inductive step:

Using the recurrence relation $u_{k+1} = u_k + u_{k-1}$,

$$u_{k+1} = \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k}{2^k\sqrt{5}} + \frac{(1+\sqrt{5})^{k-1} - (1-\sqrt{5})^{k-1}}{2^{k-1}\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^k - (1-\sqrt{5})^k + 2(1+\sqrt{5})^{k-1} - 2(1-\sqrt{5})^{k-1}}{2^k\sqrt{5}}$$

$$= \frac{\left((1+\sqrt{5})^k + 2(1+\sqrt{5})^{k-1}\right) - \left((1-\sqrt{5})^k + 2(1-\sqrt{5})^{k-1}\right)}{2^k\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^k\left(1 + \frac{2}{1+\sqrt{5}}\right) - (1-\sqrt{5})^k\left(1 + \frac{2}{1-\sqrt{5}}\right)}{2^k\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^k\left(\frac{1+\sqrt{5}}{2}\right) - (1-\sqrt{5})^k\left(\frac{1-\sqrt{5}}{2}\right)}{2^k\sqrt{5}}$$

$$= \frac{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}{2^{k+1}\sqrt{5}}$$

So true for $n = k - 1$ and $n = k \Rightarrow$ true for $n = k + 1$, and true for $n = 1$ and $n = 2$. Therefore, by induction, the closed form for the nth term of Fibonacci sequence, $u_n = \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$, is true for all $n \in \mathbb{N}$.

Watch out This is a second-order recurrence relation, so to prove it by induction you need to show that it is true for $n = 1$ **and** $n = 2$ as your basis step. You then assume it is true for $n = k - 1$ and $n = k$ and show that it is true for $n = k + 1$.

Keep in mind what you are aiming to show. In this case you need to show that $u_{k+1} = \dfrac{(1+\sqrt{5})^{k+1} - (1-\sqrt{5})^{k+1}}{2^{k+1}\sqrt{5}}$

Look for factors of $(1+\sqrt{5})$ and $(1-\sqrt{5})$.

You must write down a conclusion statement, and state that you are using mathematical induction.

Exercise 4D

(P) **1** Given that $u_{n+1} = 5u_n + 4$, with $u_1 = 4$, prove by induction that $u_n = 5^n - 1$.

(P) **2** Given that $u_{n+1} = 2u_n + 5$, with $u_1 = 3$, prove by induction that $u_n = 2^{n+2} - 5$.

(P) **3** Given that $u_{n+1} = 5u_n - 8$, with $u_1 = 3$, prove by induction that $u_n = 5^{n-1} + 2$.

(P) **4** Given that $u_{n+1} = 3u_n + 1$, with $u_1 = 1$, prove by induction that $u_n = \dfrac{3^n - 1}{2}$

(E/P) **5** A sequence $u_1, u_2, u_3, u_4, \ldots$ is defined by $u_{n+1} = \dfrac{3u_n - 1}{4}$, with $u_1 = 2$.

a Find the first four terms of the sequence. **(1 mark)**

b Prove, by induction for $n \in \mathbb{Z}^+$, that $u_n = 4\left(\frac{3}{4}\right)^n - 1$. **(5 marks)**

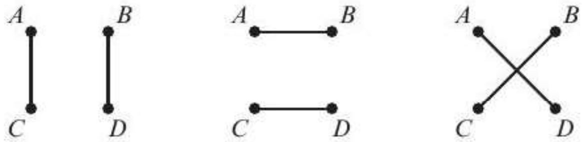
E/P **6** Given that $u_{n+1} = 4u_n - 9n$, with $u_1 = 8$, use mathematical induction to prove that $u_n = 4^n + 3n + 1$, $n \in \mathbb{Z}^+$. **(5 marks)**

E/P **7** Given that $u_{n+1} = 2n - u_n$, with $u_1 = 0$, use mathematical induction to prove that $2u_n = 2n - 1 + (-1)^n$, $n \in \mathbb{Z}^+$. **(5 marks)**

E/P **8** Given that $2u_{n+1} + u_n = 6$, with $u_1 = 4$, use mathematical induction to prove that $u_n = 2 - \left(-\frac{1}{2}\right)^{n-2}$, $n \in \mathbb{Z}^+$. **(5 marks)**

E/P **9** Given that $u_n = 3nu_{n-1}$, with $u_1 = 1$, use mathematical induction to prove that $u_n = 3^{n-1}n!$ **(5 marks)**

E/P **10** The diagram shows the three different ways that 4 people can be paired up.



Let P_n be the number of ways of pairing up a group of $2n$ people, so that $P_1 = 1$ and $P_2 = 3$.

a Explain why P_n satisfies the recurrence relation $P_n = (2n - 1)P_{n-1}$ **(3 marks)**

b Hence prove by induction that $P_n = \dfrac{(2n)!}{2^n n!}$ for all $n \in \mathbb{Z}^+$. **(5 marks)**

A **11** Given that $u_{n+2} = 5u_{n+1} - 6u_n$, with $u_1 = 1$ and $u_2 = 5$, prove by induction that $u_n = 3^n - 2^n$.

P **12** Given that $u_{n+2} = 6u_{n+1} - 9u_n$, with $u_1 = -1$ and $u_2 = 0$, prove by induction that $u_n = (n - 2)3^{n-1}$.

P **13** Given that $u_{n+2} = 7u_{n+1} - 10u_n$, with $u_1 = 1$ and $u_2 = 8$, prove by induction that $u_n = 2(5^{n-1}) - 2^{n-1}$.

P **14** Given that $u_{n+2} = 6u_{n+1} - 9u_n$, $u_1 = 3$ and $u_2 = 36$ prove by induction that $u_n = (3n - 2)3^n$.

E/P **15** A sequence $u_1, u_2, u_3, u_4, \ldots$ is defined by $u_{n+1} = 5u_n - 3(2^n)$, with $u_1 = 7$.

a Find the first four terms of the sequence. **(1 mark)**

b Prove, by induction for $n \in \mathbb{Z}^+$, that $u_n = 5^n + 2^n$. **(7 marks)**

E/P **16** The nth Lucas number L_n, is defined as follows.

$L_n = L_{n-1} + L_{n-2}$, $n \geqslant 3$, with $L_1 = 1$ and $L_2 = 3$.

Use mathematical induction to prove that the closed form for Lucas numbers is

$$L_n = \left(\frac{1 + \sqrt{5}}{2}\right)^n + \left(\frac{1 - \sqrt{5}}{2}\right)^n$$ **(7 marks)**

Mixed exercise 4

E **1** Solve the recurrence relation $u_n - 2u_{n-1} + 1 = 0$,with $u_0 = 4$. **(3 marks)**

E/P **2** **a** Solve the recurrence relation $u_n = u_{n-1} - n$, with $u_0 = 2000$. **(3 marks)**

b Hence, or otherwise, find the first negative term of the sequence. **(2 marks)**

E/P **3** **a** Solve the recurrence relation $u_n = 3u_{n-1} + 5$, with $u_0 = 0$. **(3 marks)**

b Find u_{10}. **(1 mark)**

c Find the first term of the sequence to exceed 10 million. **(2 marks)**

E/P **4** At the end of each year, a sustainable lumber company harvests 20% of its trees. To replace this stock they plant 1000 new trees. At the beginning of the first year, the company has 12 000 trees.

Let T_n represent the number of trees remaining at the end of the nth year.

a Explain why the number of trees owned by the company can be modelled by the recurrence relation $T_n = 0.8T_{n-1} + 1000$, with $T_0 = 12000$. **(3 marks)**

b Solve this recurrence relation to find a closed form for T_n. **(3 marks)**

c In the long run, how many trees can the lumber company expect to have at the end of each year? **(1 mark)**

E/P **5** At a salmon farm, the population of salmon increases by 25% each month. At the end of each month, X salmon are removed for sale.

At the beginning of the first month there are 2000 salmon in the farm.

Let S_n represent the number of salmon remaining at the end of the nth month.

a Explain why the number of salmon in the farm can be modelled by the recurrence relation $S_n = \dfrac{5S_{n-1} - 4X}{4}$, with $S_0 = 2000$. **(3 marks)**

b Prove, by induction, that $S_n = \left(\frac{5}{4}\right)^n(2000 - 4X) + 4X, n \geqslant 0$ **(5 marks)**

c Explain how the long-term population of the fish farm varies for different values of X. **(2 marks)**

E/P **6** The Smiths buy a new house in March 2018, costing £200 000. They have a deposit of £25 000. At the end of each month, interest of 0.25% is added to the balance, and the Smiths' monthly payment of £1200 is deducted from the balance.

a Write a recurrence relation showing the balance in pounds, b_n, at the end of the nth month. **(3 marks)**

b By solving your recurrence relation, determine the year in which the Smiths will pay off their mortgage. **(5 marks)**

E/P **7** The diagrams show intersecting lines drawn on a two-dimensional plane.

Assuming that all the lines are non-parallel, and that no three lines intersect at a common point, the number of points of intersection when n lines are drawn, P_n, is given by

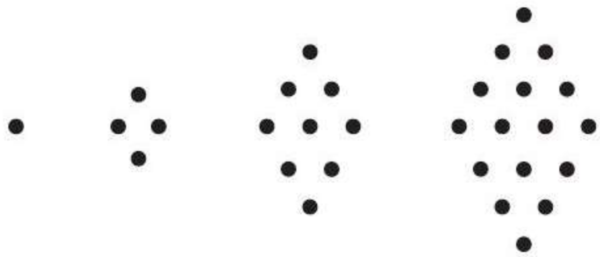
$P_1 = 0, \quad P_2 = 1, \quad P_3 = 3$

a Use the diagram above to write down the value of P_4. **(1 mark)**

b By forming and solving a suitable recurrence relation, show that $P_n = \frac{1}{2}n(n-1)$. **(4 marks)**

c Hence find the number of intersections formed when 100 such lines are drawn on the plane. **(1 mark)**

E/P **8** A sequence of patterns is formed by drawing dots in the shape of a rhombus, as shown in the diagram. The number of dots needed to draw the nth shape is represented by t_n.



a Write down the values of t_5, t_6, and t_7. **(1 mark)**

b Find in term of t_{n-1}, the recurrence relation for t_n. **(2 marks)**

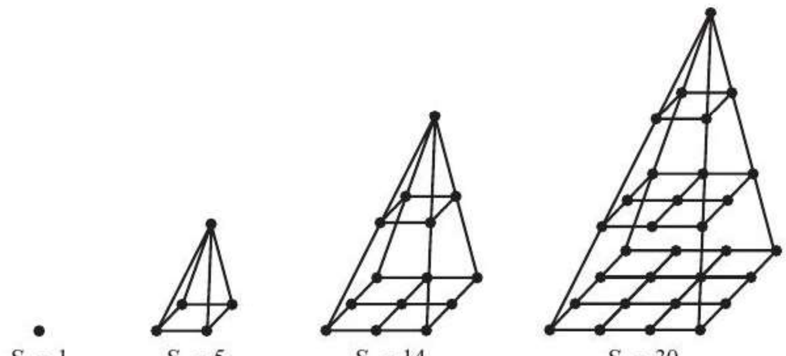
c Solve your recurrence relation for t_n, and hence determine the number of dots in the 100th pattern. **(3 marks)**

E/P **9** **a** Calculate $\begin{pmatrix} 1 & p \\ 0 & q \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}$, giving your answer as a 2×2 matrix with elements given in terms of p and q. **(3 marks)**

b Given that $\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}^n = \begin{pmatrix} 1 & a_n \\ 0 & b_n \end{pmatrix}$, write down recurrence relations for a_n in terms of a_{n-1} and b_n in terms of b_{n-1}. **(2 marks)**

c Solve your recurrence relations, and hence find $\begin{pmatrix} 1 & 4 \\ 0 & 3 \end{pmatrix}^n$, giving your answer as a 2×2 matrix with elements given in terms of n. **(4 marks)**

E/P **10** Square pyramidal numbers S_n are positive integers that can be represented by square pyramidal shapes. The first four square pyramidal numbers are 1, 5, 14, and 30, as shown in the diagram below.



a Write down S_5, S_6, and S_7. **(1 marks)**

b Find a recurrence relation for S_n in terms of S_{n-1}. **(2 marks)**

c Solve your recurrence relation to find a closed form for S_n. **(3 marks)**

E/P **11** A sequence of numbers is defined by $u_n = (n^2 + n)u_{n-1}$, with $u_1 = 1$.
Prove, by induction, that $u_n = \frac{1}{2}n!(n + 1)!$ **(5 marks)**

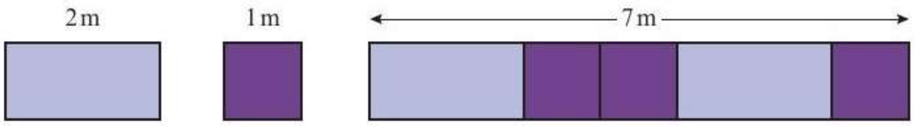
E/P **12** A sequence of numbers is defined by $u_n = (n + 2)u_{n-1}$, with $u_1 = 1$.
Prove, by induction, that $u_n = \dfrac{(n + 2)!}{6}$ **(5 marks)**

E/P **13** In a drug trial, a bacterial population is modelled as increasing at a rate of 20% each hour. A proposed antibacterial agent is introduced, and kills bacteria at a rate of $k(2^n)$ bacteria per hour, where k is a measure of the concentration of the agent.
At the beginning of the trial there are 100 bacteria present, and after n hours there are u_n bacteria present.

a Form a recurrence relation for u_n in terms of u_{n-1}, stating the initial condition. **(2 marks)**

b Show that $u_n = \left(100 + \frac{5k}{2}\right)(1.2^n) - \frac{5k}{2}(2^n)$ **(5 marks)**

A E/P **14** Flagstones come in two different sizes. Large flagstones have a length of 2 m, and small flagstones have a length of 1 m. The diagram shows a large and small flagstone, and a path of length 7 m made from a combination of these flagstones.



Let f_n represent the number of ways in which a path of length n m can be made from a combination of large and small flagstones.

a Draw the three possible paths of length 3 m. **(1 mark)**

b Explain why f_n satisfies the recurrence relation

$f_n = f_{n-1} + f_{n-2}$, with $f_1 = 1$ and $f_2 = 2$ **(3 marks)**

A path of length 200 m is to be made.

c Show that the number of ways in which this path could be constructed is given by

$$\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{201} - \left(\frac{1-\sqrt{5}}{2}\right)^{201}\right)$$ **(5 marks)**

E/P **15** A **ternary string** is a sequence of digits, where each digit can be either 0, 1 or 2.
There are 8 different ternary strings of length 2 which **do not** contain consecutive 0s.
01, 10, 02, 20, 11, 12, 21, 22
Let t_n represent the number of ternary strings of length n with no consecutive 0s.

a Find t_2 and t_3. **(1 mark)**

b Explain why t_n satisfies the recurrence relation $t_n = 2t_{n-1} + 2t_{n-2}$. **(3 marks)**

c Find t_6. **(1 mark)**

d Find:

i a closed form for t_n in terms of n

ii the number of different ternary strings length 15 which do not contain consecutive 0s. **(5 marks)**

E **16** **a** Find the general solution to the recurrence relation

$u_{n+2} = u_{n+1} + 2u_n, n \geqslant 1$ **(3 marks)**

b Given that $u_1 = 1$ and $u_2 = 2$, find the particular solution to the recurrence relation. **(3 marks)**

E/P **17** **a** Find the general solution to the recurrence relation

$x_{n+2} = 7x_{n+1} - 10x_n + 3, n \geqslant 1$ **(4 marks)**

b Given that $x_1 = 1$ and $x_2 = 2$, find the particular solution to the recurrence relation. **(3 marks)**

A **18** Solve the recurrence relation $a_n = 2a_{n-1} + 15a_{n-2} + 2^n$, with $a_1 = 2$ and $a_2 = 4$. **(8 marks)**

E/P

E/P **19** A sequence satisfies the recurrence relation $u_n = -2nu_{n-1} + 3n(n-1)u_{n-2}$, with $u_0 = 1$ and $u_1 = 2$.
Prove, by induction, that a closed form for this sequence is $u_n = \frac{n!}{4}(5 - (-3)^n)$ **(7 marks)**

E/P **20** **a** Find a closed form for the sequence defined by the recurrence relation
$u_n = \sqrt{2}u_{n-1} - u_{n-2}$, with $u_0 = u_1 = 1$ **(5 marks)**

b Hence show that the sequence is periodic and state its period. **(3 marks)**

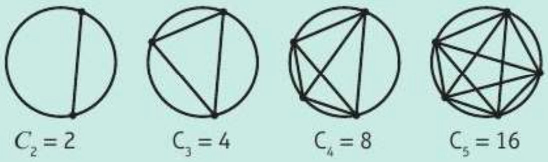
E/P **21** Messages are transmitted over a network using two types of signal packet. Type A signal packets require 1 microsecond to transmit, and type B signal packets require 2 microseconds to transmit. The packets are transmitted consecutively with no gaps between them.
The number of different messages consisting of sequences of these two types of signal packet that can be sent in n microseconds is denoted by S_n.

a Write a recurrence relation for S_{n+2} in terms of S_{n+1} and S_n. State the initial conditions for your recurrence relation. **(3 marks)**

b Solve your recurrence relation to find an expression for S_n in terms of n. **(5 marks)**

Challenge

1 The circles in the diagram have been subdivided into regions by drawing all possible chords between points drawn on the circumference of the circle. Let C_n denote the maximum number of regions that are formed in this way when n points are drawn on the circumference of a circle.



a Find C_6.

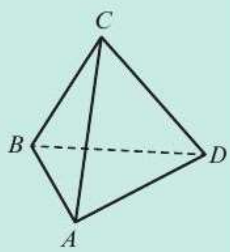
b Explain why it is always possible to choose a new point on the circumference of the circle such that all the new chords drawn from that point do not intersect the existing chords at any existing points of intersection.

c Find, in terms of C_{n-1} and n, a recurrence relation for C_n.

d Solve your recurrence relation, and hence determine the maximum number of regions created by 100 points.

2 The diagram shows a tetrahedron $ABCD$.
A spider walks along the edges of the tetrahedron, starting and ending at vertex A. A walk of length n traverses exactly n edges, so that there are three possible walks of length 2:

$A \to B \to A$
$A \to C \to A$
$A \to D \to A$



a Explain why there are no possible walks of length 1.

b Find the number of possible walks of length 3.

c By formulating and solving a suitable recurrence relation find a closed form for the total number of possible walks of length n.

Summary of key points

1 A recurrence relation is an equation that defines a sequence based on a rule that gives the each term as a function of the previous term(s).

2 The **order** of a recurrence relation is the difference between the highest and lowest subscript in the relation.

3 A sequence u_n is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation. It is also called the **closed form** of the sequence.

4 A **first-order** recurrence relation is one in which u_n can be given as a function of n and u_{n-1} only.

5 A **first-order linear** recurrence relation can be written in the form $\boldsymbol{u_n = au_{n-1} + \mathrm{g}(n)}$.

- If $g(n) = 0$, then the equation is homogeneous.
- The solution to the first-order homogeneous linear recurrence relation $u_n = au_{n-1}$ is given by $u_n = u_0a^n$ or $u_n = u_1a^{n-1}$.
- The solution to the first-order non-homogeneous linear recurrence relation $u_n = u_{n-1} + \mathrm{g}(n)$ is given by $u_n = u_0 + \sum_{r=1}^{n}\mathrm{g}(r)$.
- When solving a recurrence relation of the form $u_n = au_{n-1} + \mathrm{g}(n)$, the form of the particular solution will depend on $\mathrm{g}(n)$:

Form of $\mathrm{g}(n)$	Form of particular solution
p with $a \neq 1$	λ
$pn + q$ with $a \neq 1$	$\lambda n + \mu$
kp^n with $p \neq a$	λk^n
ka^n	λna^n

6 To solve the recurrence relation $\boldsymbol{u_n = au_{n-1} + \mathrm{g}(n)}$,

- Find the complementary function, which is the general solution to the associated homogeneous recurrence relation $u_n = au_{n-1}$.
- Choose an appropriate form for a particular solution then substitute into the original recurrence relation to find the values of any coefficients.
- The general solution is un = C.F. + P.S. = ca^n + P.S.
- Use the initial condition to find the value of the arbitrary constant.

7 If $u_n = \mathrm{F}(n)$ and $u_n = \mathrm{G}(n)$ are particular solutions to a linear recurrence relation, then $u_n = a\mathrm{F}(n) + b\mathrm{G}(n)$, where a and b are constants, is also a solution.

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8 A **second-order linear** recurrence relation can be written in the form $\boldsymbol{u_n = au_{n-1} + bu_{n-2} + \mathrm{g}(n)}$, where a and b are real constants.

- If $\mathrm{g}(n) = 0$, then the equation is homogeneous.

9 You can find a general solution to a **second-order homogeneous** linear recurrence relation, $u_n = au_{n-1} + bu_{n-2}$, by considering the auxiliary equation, $r^2 - ar - b = 0$.
You need to consider three different cases:

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Case 1: Distinct real roots
If the auxiliary equation has distinct real roots α and β, then the general solution will have the form $u_n = A\alpha^n + B\beta^n$ where A and B are arbitrary constants.

Case 2: Repeated root
If the auxiliary equation has a repeated real root α, then the general solution will have the form $u_n = (A + Bn)\alpha^n$ where A and B are arbitrary constants.

Case 3: Complex roots
If the auxiliary equation has two complex roots $\alpha = re^{i\theta}$ and $\beta = re^{-i\theta}$, then the general solution will have the form $u_n = r^n(A\cos n\theta + B\sin n\theta)$, or $u_n = A\alpha^n + B\beta^n$, where A and B are arbitrary constants.

10 To solve the recurrence relation $\boldsymbol{u_n = au_{n-1} + bu_{n-2} + g(n)}$,

- Find the complementary function, which is the general solution to the associated homogeneous recurrence relation $u_n = au_{n-1} + bu_{n-2}$.
- Choose an appropriate form for a particular solution then substitute into the original recurrence relation to find the values of any coefficients.
- The general solution is u_n = C.F. + P.S.
- Use the initial conditions to find the values of the arbitrary constants.

11 For the recurrence relation $u_n = au_{n-1} + bu_{n-2} + g(n)$, with auxiliary equation with roots α and β, try the following forms for a particular solution:

Form of g(n)	Form of particular solution
p with $\alpha, \beta \neq 1$	λ
$pn + q$, with $\alpha, \beta \neq 1$	$\lambda n + \mu$
kp^n with $p \neq \alpha, \beta$	λp^n
p with $\alpha = 1, \beta \neq 1$	λn
$pn + q$ with $\alpha = 1, \beta \neq 1$	$\lambda n^2 + \mu n$
p with $\alpha = \beta = 1$	λn^2
$pn + q$ with $\alpha = \beta = 1$	$\lambda n^3 + \mu n^2$
$k\alpha^n$ with $\alpha \neq \beta$	$\lambda n\alpha^n$
$k\alpha^n$ with $\alpha = \beta$	$\lambda n^2\alpha^n$

12 You can prove that a closed form satisfies a given recurrence relation using mathematical induction.

13 When you are proving the closed form of a second-order recurrence relation by mathematical induction, you need to:

- show that the closed form is true for two consecutive values of n (basis step)
- assume that the closed form is true for $n = k$ and $n = k - 1$ (assumption step), then show that it is true for $n = k + 1$ (inductive step).