# Group Theory – Post 2018 Mix

1.

The set $M$ contains all matrices of the form $\mathbf{X}^n$, where $\mathbf{X} = \frac{1}{\sqrt{3}}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$ and $n$ is a positive integer.

**(i)** Show that $M$ contains exactly 12 elements. **[4]**

**(ii)** Deduce that $M$, together with the operation of matrix multiplication, form a cyclic group $G$. **[5]**

**(iii)** Determine all the proper subgroups of $G$. **[5]**

2.

A class of students is set the task of finding a group of functions, under composition of functions, of order 6.

Student P suggests that this can be achieved by finding a function f for which $f^6(x) = x$ and using this as a generator for the group.

**(i)** Explain why the suggestion by Student P might not work. **[2]**

Student Q observes that their class has already found a group of order 6 in a previous task; a group consisting of the powers of a particular, non-singular $2 \times 2$ real matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, under the operation of matrix multiplication.

**(ii)** Explain why such a group is only possible if $\det(\mathbf{M}) = 1$ or $-1$. **[2]**

**(iii)** Write down values of $a$, $b$, $c$ and $d$ that would give a suitable matrix $\mathbf{M}$ for which $\mathbf{M}^6 = \mathbf{I}$ and $\det(\mathbf{M}) = 1$. **[1]**

Student Q believes that it is possible to construct a rational function f in the form $f(x) = \frac{ax+b}{cx+d}$ so that the group of functions is isomorphic to the matrix group which is generated by the matrix $\mathbf{M}$ of part **(iii)**.

**(iv) (a)** Write down and simplify the function f that, according to Student Q, corresponds to $\mathbf{M}$. **[1]**

**(b)** By calculating $\mathbf{M}^3$, show that Student Q's suggestion does not work. **[2]**

**(c)** Find a different function f that will satisfy the requirements of the task. **[4]**

3.

The set $L$ consists of all points $(x, y)$ in the cartesian plane, with $x \neq 0$. The operation $\diamond$ is defined by $(a, b) \diamond (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in L$.

**(a)** **(i)** Show that $L$ is closed under $\diamond$. **[1]**

**(ii)** Prove that $\diamond$ is associative on $L$. **[4]**

**(iii)** Find the identity element of $L$ under $\diamond$. **[2]**

**(iv)** Find the inverse element of $(a, b)$ under $\diamond$. **[3]**

**(b)** Find a subgroup of $(L, \diamond)$ of order 2. **[2]**

4.

The set $X$ consists of all $2 \times 2$ matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$, where $x$ and $y$ are real numbers which are not **both** zero.

**(i)** **(a)** The matrices $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $\begin{pmatrix} c & -d \\ d & c \end{pmatrix}$ are both elements of $X$.

Show that $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$ for some real numbers $p$ and $q$ to be found in terms of $a$, $b$, $c$ and $d$. **[2]**

**(b)** Prove by contradiction that $p$ and $q$ are not **both** zero. **[5]**

**(ii)** Prove that $X$, under matrix multiplication, forms a group $G$.
[You may use the result that matrix multiplication is associative.] **[4]**

**(iii)** Determine a subgroup of $G$ of order 17. **[2]**

5.

The group $G$ consists of a set $S$ together with $\times_{80}$, the operation of multiplication modulo 80. It is given that $S$ is the smallest set which contains the element 11.

**(a)** By constructing the Cayley table for $G$, determine all the elements of $S$. **[5]**

The Cayley table for a second group, $H$, also with the operation $\times_{80}$, is shown below.

| $\times_{80}$ | 1 | 9 | 31 | 39 |
|---|---|---|---|---|
| 1 | 1 | 9 | 31 | 39 |
| 9 | 9 | 1 | 39 | 31 |
| 31 | 31 | 39 | 1 | 9 |
| 39 | 39 | 31 | 9 | 1 |

**(b)** Use the two Cayley tables to explain why $G$ and $H$ are not isomorphic. **[2]**

**(c)** **(i)** List

- all the proper subgroups of $G$,
- all the proper subgroups of $H$. **[3]**

**(ii)** Use your answers to **(c)** **(i)** to give another reason why $G$ and $H$ are not isomorphic. **[1]**

6.

The following Cayley table is for a set $\{a, b, c, d\}$ under a suitable binary operation.

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $b$ | | $a$ | |
| $b$ | | | | |
| $c$ | | | $c$ | |
| $d$ | $d$ | | | $a$ |

**(a)** Given that the Latin square property holds for this Cayley table, complete it using the table supplied in the Printed Answer Booklet. **[4]**

**(b)** Using your completed Cayley table, explain why the set does **not** form a group under the binary operation. **[1]**

7.

The group $G$ consists of the set $\{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\}$ under $\times_{39}$, the operation of multiplication modulo 39.

**(a)** List the possible orders of proper subgroups of $G$, justifying your answer. **[2]**

**(b)** List the elements of the subset of $G$ generated by the element 3. **[1]**

**(c)** State the identity element of $G$. **[1]**

**(d)** Determine the order of the element 18. **[2]**

**(e)** Find the two elements $g_1$ and $g_2$ in $G$ which satisfy $g \times_{39} g = 3$. **[3]**

The group $H$ consists of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ under $\times_{13}$, the operation of multiplication modulo 13. You are given that $G$ is isomorphic to $H$.

A student states that $G$ is isomorphic to $H$ because each element $3x$ in $G$ maps directly to the element $x$ in $H$ (for $x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$).

**(f)** Explain why this student is incorrect. **[1]**

8.

The following Cayley table is for $G$, a group of order 6. The identity element is $e$ and the group is generated by the elements $a$ and $b$.

| $G$ | $e$ | $a$ | $a^2$ | $b$ | $ab$ | $a^2b$ |
|---|---|---|---|---|---|---|
| $e$ | $e$ | $a$ | $a^2$ | $b$ | $ab$ | $a^2b$ |
| $a$ | $a$ | $a^2$ | $e$ | $ab$ | $a^2b$ | $b$ |
| $a^2$ | $a^2$ | $e$ | $a$ | $a^2b$ | $b$ | $ab$ |
| $b$ | $b$ | $a^2b$ | $ab$ | $e$ | $a^2$ | $a$ |
| $ab$ | $ab$ | $b$ | $a^2b$ | $a$ | $e$ | $a^2$ |
| $a^2b$ | $a^2b$ | $ab$ | $b$ | $a^2$ | $a$ | $e$ |

**(a)** List all the proper subgroups of $G$. **[4]**

**(b)** State another group of order 6 to which $G$ is isomorphic. **[1]**

9.

The binary operation $\Diamond$ is defined on the set $\mathbb{C}$ of complex numbers by

$(a+\mathrm{i}b)\Diamond(c+\mathrm{i}d) = ac+\mathrm{i}(b+ad)$

where $a$, $b$, $c$ and $d$ are real numbers.

**(a)** Is $\mathbb{C}$ closed under $\Diamond$? Justify your answer. **[1]**

**(b)** Prove that $\Diamond$ is associative on $\mathbb{C}$. **[4]**

**(c)** Determine the identity element of $\mathbb{C}$ under $\Diamond$. **[2]**

**(d)** Determine the largest subset S of $\mathbb{C}$ such that (S, $\Diamond$) is a group. **[3]**

10.

**(a)** Explain why all groups of even order must contain at least one self-inverse element (that is, an element of order 2). **[2]**

**(b)** Prove that any group, in which every (non-identity) element is self-inverse, is abelian. **[2]**

**(c)** A student believes that, if $x$ and $y$ are two distinct, non-identity, self-inverse elements of a group, then the element $xy$ is also self-inverse.

The table shown here is the Cayley table for the non-cyclic group of order 6, having elements $i$, $a$, $b$, $c$, $d$ and $e$, where $i$ is the identity.

| | $i$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|---|
| $i$ | $i$ | $a$ | $b$ | $c$ | $d$ | $e$ |
| $a$ | $a$ | $i$ | $d$ | $e$ | $b$ | $c$ |
| $b$ | $b$ | $e$ | $i$ | $d$ | $c$ | $a$ |
| $c$ | $c$ | $d$ | $e$ | $i$ | $a$ | $b$ |
| $d$ | $d$ | $c$ | $a$ | $b$ | $e$ | $i$ |
| $e$ | $e$ | $b$ | $c$ | $a$ | $i$ | $d$ |

By considering the elements of this group, produce a counter-example which proves that this student is wrong. **[2]**

**(d)** A group $G$ has order $4n+2$, for some positive integer $n$, and $i$ is the identity element of $G$. Let $x$ and $y$ be two distinct, non-identity, self-inverse elements of $G$. By considering the set $H=\{i, x, y, xy\}$, prove by contradiction that not all elements of $G$ are self-inverse. **[4]**

11.

The group $G$ consists of the set $S = \{1, 9, 17, 25\}$ under $\times_{32}$, the operation of multiplication modulo 32.

**(i)** Complete the Cayley table for $G$ given in the Printed Answer Booklet. **[2]**

**(ii)** Up to isomorphisms, there are only two groups of order 4.

- $C_4$, the cyclic group of order 4
- $K_4$, the non-cyclic (Klein) group of order 4

State, with justification, to which of these two groups $G$ is isomorphic. **[2]**

12.

The set $C$ consists of the set of all complex numbers excluding 1 and $-1$. The operation $\oplus$ is defined on the elements of $C$ by $a \oplus b = \dfrac{a+b}{ab+1}$ where $a, b \in C$.

**(a)** Determine the identity element of $C$ under $\oplus$. **[2]**

**(b)** For each element $x$ in $C$ show that it has an inverse element in $C$. **[2]**

**(c)** Show that $\oplus$ is associative on $C$. **[3]**

**(d)** Explain why $(C, \oplus)$ is not a group. **[1]**

**(e)** Find a subset, $D$, of $C$ such that $(D, \oplus)$ is a group of order 3. **[3]**

13.

(i) A binary operation * is defined on positive real numbers by

$$a * b = a + b + ab$$

Prove that the operation * is associative.

**(4)**

(ii) The set $G = \{1, 2, 3, 4, 5, 6\}$ forms a group under the operation of multiplication modulo 7

(a) Show that $G$ is cyclic.

**(2)**

The set $H = \{1, 5, 7, 11, 13, 17\}$ forms a group under the operation of multiplication modulo 18

(b) List all the subgroups of $H$.

**(3)**

(c) Describe an isomorphism between $G$ and $H$.

**(3)**

14.

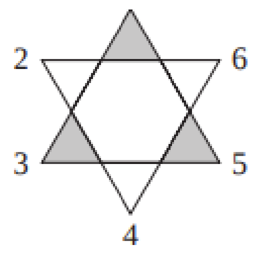


**Figure 3**

Figure 3 shows a plane shape made up of a regular hexagon with an equilateral triangle joined to each edge and with alternate equilateral triangles shaded.

The symmetries of this shape are the rotations and reflections of the plane that preserve the shape and its shading.

The symmetries of the shape can be represented by permutations of the six vertices labelled 1 to 6 in Figure 3. The set of these permutations with the operation of composition form a group, $G$.

(a) Describe geometrically the symmetry of the shape represented by the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$$

**(2)**

(b) Write down, in similar two-line notation, the remaining elements of the group $G$.

**(4)**

(c) Explain why each of the following statements is false, making your reasoning clear.

(i) $G$ has a subgroup of order 4

(ii) $G$ is cyclic.

**(2)**

Diagram 1, on page 23, shows an unshaded shape with the same outline as the shape in Figure 3.

(d) Shade the shape in Diagram 1 in such a way that the group of symmetries of the resulting shaded shape is isomorphic to the cyclic group of order 6

**(2)**

15.

A binary operation $\star$ on the set of non-negative integers, $\mathbb{Z}_0^+$, is defined by

$$m \star n = |m - n| \qquad m, n \in \mathbb{Z}_0^+$$

(a) Explain why $\mathbb{Z}_0^+$ is closed under the operation $\star$

**(1)**

(b) Show that 0 is an identity for $(\mathbb{Z}_0^+, \star)$

**(2)**

(c) Show that all elements of $\mathbb{Z}_0^+$ have an inverse under $\star$

**(2)**

(d) Determine if $\mathbb{Z}_0^+$ forms a group under $\star$, giving clear justification for your answer.

**(3)**

16.

The group $S_4$ is the set of all possible permutations that can be performed on the four numbers 1, 2, 3 and 4, under the operation of composition.

For the group $S_4$

(a) write down the identity element,

**(1)**

(b) write down the inverse of the element $a$, where

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

**(1)**

(c) demonstrate that the operation of composition is associative using the following elements

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix} \quad \text{and } c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

**(2)**

(d) Explain why it is possible for the group $S_4$ to have a subgroup of order 4
You do not need to find such a subgroup.

**(2)**

17.

The set $G = \mathbb{R} - \left\{-\frac{3}{2}\right\}$ with the operation of $x \bullet y = 3(x + y + 1) + 2xy$ forms a group.

(a) Determine the identity element of this group.

**(2)**

(b) Determine the inverse of a general element $x$ in this group.

**(3)**

(c) Explain why the value $-\frac{3}{2}$ must be excluded from $G$ in order for this to be a group.

**(1)**

18.

(i) A group $G$ contains distinct elements $a$, $b$ and $e$ where $e$ is the identity element and the group operation is multiplication.

Given $a^2b = ba$, prove $ab \neq ba$

(4)

(ii) The set $H = \{1, 2, 4, 7, 8, 11, 13, 14\}$ forms a group under the operation of multiplication modulo 15

(a) Find the order of each element of $H$.

(3)

(b) Find three subgroups of $H$ each of order 4, and describe each of these subgroups.

(4)

The elements of another group $J$ are the matrices $\begin{pmatrix} \cos\left(\frac{k\pi}{4}\right) & \sin\left(\frac{k\pi}{4}\right) \\ -\sin\left(\frac{k\pi}{4}\right) & \cos\left(\frac{k\pi}{4}\right) \end{pmatrix}$

where $k = 1, 2, 3, 4, 5, 6, 7, 8$ and the group operation is matrix multiplication.

(c) Determine whether $H$ and $J$ are isomorphic, giving a reason for your answer.

(2)