Multi-Variable Calculus – Post 2018 Mix

1.

The surface S has equation $z = \frac{x}{y}\sin y + \frac{y}{x}\cos x$ where $0 < x \leqslant \pi$ and $0 < y \leqslant \pi$.

(i) Find

- $\frac{\partial z}{\partial x}$,
- $\frac{\partial z}{\partial y}$. **[4]**

(ii) Determine the equation of the tangent plane to S at the point A where $x = y = \frac{1}{4}\pi$. Give your answer in the form $ax + by + cz = d$ where a, b, c and d are exact constants. **[5]**

(iii) Write down a normal vector to S at A. **[1]**

2.

The function $w = \mathrm{f}(x, y, z)$ is given by $\mathrm{f}(x, y, z) = x^2yz + 2xy^2z + 3xyz^2 - 24xyz$, for $x, y, z \neq 0$.

(i) (a) Find

- f_x,
- f_y,
- f_z. **[4]**

(b) Hence find the values of a, b, c and d for which w has a stationary value when $d = \mathrm{f}(a, b, c)$. **[5]**

(ii) You are given that this stationary value is a local minimum of w. Find values of x, y and z which show that it is not a global minimum of w. **[2]**

3.

A surface has equation $z = x\tan y$ for $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$.

(a) Find

- $\frac{\partial z}{\partial x}$,
- $\frac{\partial z}{\partial y}$. **[2]**

(b) Find in cartesian form, the equation of the tangent plane to the surface at the point where $x = 1$ and $y = \frac{1}{4}\pi$. **[5]**

4.

For each value of t, the surface S_t has equation $z = tx^2 + y^2 + 3xy - y$.

(a) Verify that there are no stationary points on S_t when $t = \frac{9}{4}$. **[4]**

(Part b not on our spec)

5.

Given $z = x\sin y + y\cos x$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} + z = 0$. **[5]**

6.

A surface S has equation $z = \mathrm{f}(x, y)$, where $\mathrm{f}(x, y) = 2x^2 - y^2 + 3xy + 17y$. It is given that S has a single stationary point, P.

(i) (a) Determine the coordinates of P. **[5]**

~~**(b)** Determine the nature of P. **[3]**~~

(ii) Find the equation of the tangent plane to S at the point $Q(1, 2, 38)$. **[2]**

7.

A surface has equation $z = \mathrm{f}(x, y)$ where $\mathrm{f}(x, y) = x^2 \sin y + 2y \cos x$.

(a) Determine f_x , f_y , f_{xx} , f_{yy} , f_{xy} and f_{yx} . **[5]**

(b) **(i)** Verify that z has a stationary point at $\left(\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{1}{4}\pi^2\right)$. **[3]**

8.

For $x,\ y \in \mathbb{R}$, the function f is given by $\mathrm{f}(x, y) = 2x^2y^7 + 3x^5y^4 - 5x^8y$.

(a) Prove that $x\mathrm{f}_x + y\mathrm{f}_y = n\mathrm{f}$, where n is a positive integer to be determined. **[5]**

(b) Show that $x\mathrm{f}_{xx} + y\mathrm{f}_{xy} = (n-1)\mathrm{f}_x$. **[4]**

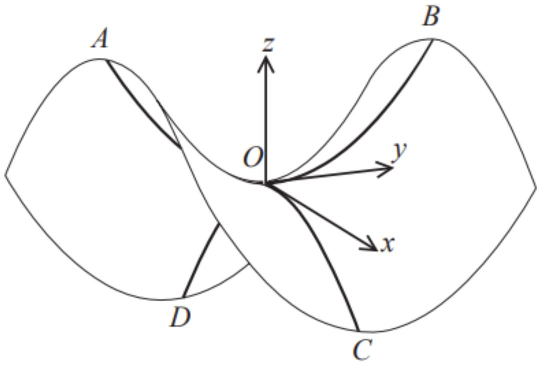
9.

The surface S has equation $x^2 + y^2 + z^2 = xyz - 1$.

(a) Show that $(2z - xy)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = 2(1 + z^2)$. **[6]**

(b) Deduce that S has no stationary point. **[2]**

10.



A student wishes to model the saddle of a horse. They use a surface described by a function of the form $z = \mathrm{f}(x, y)$ with a saddle point at the origin O. The z-axis is vertically upwards. The x- and y-axes lie in a horizontal plane, with the x-axis across the horse and the y-axis along the length of the horse (see diagram).

The arc AOB is part of a parabola which lies in the yz-plane. The arc COD is part of a parabola which lies in the xz-plane. The saddle is symmetric in both the xz-plane and yz-plane.

The length of the saddle, the distance AB, is to be 0.6 m with both A and B at a height of 0.27 m above O. The width of the saddle, the distance CD, is to be 0.5 m with both C and D at a depth of 0.4 m below O.

(a) On separate diagrams, sketch the sections $x = 0$ and $y = 0$. **[2]**

(b) Determine a function f that describes the saddle. [You do not need to state the domain of function f.] **[5]**

11.

The surface E has equation $z = \sqrt{500 - 3x^2 - 2y^2}$.

(a) Determine the values of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point P on E with coordinates (11, −8, 3). **[4]**

(b) Find the equation of the tangent plane to E at P, giving your answer in the form $ax + by + cz = d$ where a, b, c and d are integers. **[2]**

12.

For all real values of x and y the surface S has equation $z = 4x^2 + 4xy + y^2 + 6x + 3y + k$, where k is a constant and an integer.

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. **[2]**

(b) Determine the smallest value of the integer k for which the whole of S lies above the x-y plane. **[7]**

13.

The surface S is defined for all real x and y by the equation $z = x^2 + 2xy$. The intersection of S with the plane Π gives a section of the surface. On the axes provided in the Printed Answer Booklet, sketch this section when the equation of Π is each of the following.

(a) $x = 1$ **[2]**

(b) $y = 1$ **[2]**