Multi-Variable Calculus – Post 2018 Mix – Mark Scheme

1.

(i)	$\frac{\partial z}{\partial x} = \frac{1}{y}\sin y - \frac{y}{x}\sin x - \frac{y}{x^2}\cos x$ **oe**	M1 A1	1.1a 1.1	Partial differentiation w.r.t x (including use of the *Product* or *Quotient Rule*)
	$\frac{\partial z}{\partial y} = \frac{x}{y}\cos x - \frac{x}{y^2}\sin y + \frac{1}{x}\cos x$ **oe**	M1 A1 [4]	1.1a 1.1	Partial differentiation w.r.t y (including use of the *Product* or *Quotient Rule*)
(ii)	When $x = y = \frac{1}{4}\pi$, $z = \sqrt{2}$	B1	1.1	
	$\frac{\partial z}{\partial x} = \frac{-1}{\sqrt{2}}$ and $\frac{\partial z}{\partial y} = \frac{1}{\sqrt{2}}$ **oe**	B1 B1	1.1 1.1	
	Eqn. of tangent-plane is $z = \frac{-1}{\sqrt{2}}(x - \frac{1}{4}\pi) + \frac{1}{\sqrt{2}}(y - \frac{1}{4}\pi) + \sqrt{2}$	M1	2.2a	
	$\Rightarrow x - y + z\sqrt{2} = 2$ **oe**	A1 [5]	1.1	
(iii)	$\begin{pmatrix} 1 \\ -1 \\ \sqrt{2} \end{pmatrix}$ or any suitable multiple	B1 [1]	2.2a	**FT** from their answer to **(ii)**

2.

(i)	(a)	$f_x = 2xyz + 2y^2z + 3yz^2 - 24yz$ or $yz(2x + 2y + 3z - 24)$ *	M1 A1	1.1a 1.2	Good attempt at (at least) one partial derivative
		$f_y = x^2z + 4xyz + 3xz^2 - 24xz$ or $xz(x + 4y + 3z - 24)$	A1	1.1	
		$f_z = x^2y + 2xy^2 + 6xyz - 24xy$ or $xy(x + 2y + 6z - 24)$	A1 [4]	1.1	
	(b)	Setting all p.d.s to zero	M1	1.1a	
		All three are zero ($x, y, z \neq 0$) when $2x + 2y + 3z = 24$, $x + 4y + 3z = 24$, $x + 2y + 6z = 24$	M1	2.1	Factoring out the non-zero terms and setting up a solvable system of equations
		$x = 6,\ y = 3,\ z = 2$	M1 A1	1.1a 1.1	Solving a 3×3 system of equations (**BC**, for instance) Correct
		$w = -216$	A1 [5]	1.1	**cao** (no need to mention these are a, b, c, d)
(ii)		Want (e.g.) x large and $-_{ve}$, y small and $-_{ve}$, z small and $+_{ve}$	M1	3.1a	Correct idea
		e.g. $x = -10,\ y = -1,\ z = 1$ so that $w = -330 < -216$	A1 [2]	3.2a	Demonstrated correctly

3.

(a)	$\frac{\partial z}{\partial x} = \tan y$ and $\frac{\partial z}{\partial y} = x\sec^2 y$	B1 B1 [2]	1.1 1.1	
(b)	$x = 1, y = \frac{1}{4}\pi$ substituted into the two partial derivatives	M1	1.1a	
	$\frac{\partial z}{\partial x} = 1$ and $\frac{\partial z}{\partial y} = 2$	A1	1.1	Both
	$z = 1$	B1	1.1	
	Equation of tangent plane is $z - 1 = 1(x - 1) + 2(y - \frac{1}{4}\pi)$	M1FT	1.1a	**FT** three previous values
	i.e. $x + 2y - z = \frac{1}{2}\pi$ oe	A1 [5]	1.1	

4.

$z = tx^2 + y^2 + 3xy - y$			
$\frac{\partial z}{\partial x} = 2tx + 3y \qquad \frac{\partial z}{\partial y} = 2y + 3x - 1$	B1 B1	3.1a 1.1	Finding the coordinates of the stationary points is not required.
When $t = \frac{9}{4}$, if there is a stationary point then			
$\frac{\partial z}{\partial x} = \frac{9}{2}x + 3y = 0$ and $\frac{\partial z}{\partial y} = 2y + 3x - 1 = 0$	M1	3.1a	
$2\frac{\partial z}{\partial x} = 9x + 6y = 0$			
$3\frac{\partial z}{\partial y} = 6y + 9x - 3 = 0$ which is a contradiction, so there is no stationary point on S when $t = \frac{9}{4}$.	E1	3.2a	
	[4]		

5.

$\frac{\partial z}{\partial x} = \sin y - y\cos x \qquad \frac{\partial z}{\partial y} = x\cos y + \cos x$	M1	2.1	Attempt at 1st and 2nd partial derivatives of z with respect to x and y; at least one correct
	A1	1.1	
$\frac{\partial^2 z}{\partial x^2} = -y\cos x \qquad \frac{\partial^2 z}{\partial y^2} = -x\sin y$	A1 FT A1 FT	1.1 1.1	FT for each correct second partial derivative
$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + z$ $= -y\cos x - x\sin y + (x\sin y + y\cos x) = 0$	E1	2.1	AG Shown clearly
	[5]		

6.

(i)	(a)	$f_x = 4x + 3y$ and	B1	1.1	
		$f_y = -2y + 3x + 17$	B1	1.1	
		Both are zero when			
		$4x + 3y = 0$ and $-2y + 3x + 17 = 0$	M1	2.1	The connection must be made explicitly
		$x = -3,\ y = 4$	A1	1.1	BC
		$z = 34$	A1	1.1	
			[5]		
(ii)		$z = f(a, b) + (x-a)f_x(a, b)$ $+(y-b)f_y(a, b)$	M1	1.1	Quoting correct form, including attempt to substitute in values
		i.e. $z = 38 + 10(x-1) + 16(y-2)$ or $10x + 16y - z = 4$	A1	1.1	Any correct simplified form
			[2]		

7.

(a)		$f_x = 2x\sin y - 2y\sin x \qquad f_y = x^2\cos y + 2\cos x$	1.1a 1.1	B1 B1	
		$f_{xx} = 2\sin y - 2y\cos x \qquad f_{yy} = -x^2\sin y$	1.1 2.5	B1 B1	
		$f_{xy} = f_{yx} = 2x\cos y - 2\sin x$	1.1	B1	Both must be written down, or stated equal
				[5]	
(b)	i	When $x = y = \frac{1}{2}\pi$, $f_x = f_y = 0$	1.1a	M1	Allow statement only
		$\Rightarrow$ stationary point	2.2a	A1	
		Visible check/calculation that $z = \left(\frac{1}{2}\pi\right)^2 \times 1 + 2 \times \left(\frac{1}{2}\pi\right) \times 0 = \frac{1}{4}\pi^2$	2.1	B1	
				[3]	

8.

(a)		$f_x = 4xy^7 + 15x^4y^4 - 40x^7y$ and $f_y = 14x^2y^6 + 12x^5y^3 - 5x^8$	M1 A1 A1	1.1 1.1 1.1	Clear attempt at partial differentiation
		Then $x f_x + y f_y = 4x^2y^7 + 15x^5y^4 - 40x^8y + 14x^2y^7 + 12x^5y^4 - 5x^8y$	M1	2.1	
		$= 18x^2y^7 + 27x^5y^4 - 45x^8y = 9\,f$	A1	1.1	$n = 9$ must be clearly stated
			[5]		
(b)		$f_{xx} = 4y^7 + 60x^3y^4 - 280x^6y$ and $f_{xy} = 28xy^6 + 60x^4y^3 - 40x^7$	B1 B1	1.1 1.1	
		Then $x f_{xx} + y f_{xy} = 4xy^7 + 60x^4y^4 - 280x^7y + 28xy^7 + 60x^4y^4 - 40x^7y$	M1	2.1	
		$= 32xy^7 + 120x^4y^4 - 320x^7y = 8\,f_x$	A1	1.1	
		Alternative method Differentiate. **(a)**'s result w.r.t. x: $x f_{xx} + f_x + y f_{yx} = n f_x$ **M1 A1** $f_{yx} = f_{xy}$ (by the Mixed Derivative Theorem) **B1** $\Rightarrow x f_{xx} + y f_{xy} = (n-1) f_x$ **A1**			
			[4]		

9.

(a)		Differentiating $x^2 + y^2 + z^2 = xyz - 1$ partially w.r.t. either x or y	M1	1.1	Attempt with LHS correct
		x): $2x + 2z\frac{\partial z}{\partial x} = xy\frac{\partial z}{\partial x} + yz$ ❶ y): $2y + 2z\frac{\partial z}{\partial y} = xy\frac{\partial z}{\partial y} + xz$ ❷	A1 A1	1.1 1.1	First correct Correct or **FT** $x \leftrightarrow y$ (by symmetry)
		x.❶ + y.❷ $\Rightarrow 2(x^2+y^2) + 2z\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = xy\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) + 2xyz$	M1	3.1a	$\Rightarrow (2z - xy)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = 2(xyz - x^2 - y^2)$
		Now $x^2 + y^2 + z^2 = xyz - 1 \Rightarrow xyz - x^2 - y^2 = z^2 + 1$	M1	1.1	
		giving $(2z - xy)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = 2(1 + z^2)$	A1	2.2a	**AG** legitimately obtained
			[6]		
(b)		If both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are zero …	M1	2.1	Considering conditions for stationary points
		… then LHS = 0 while RHS ≥ 2 (i.e. > 0) (⇒⇐)	A1	2.4	Contradiction justified
			[2]		

10.

(a)		The section $x = 0$, i.e. $z = f(0, y)$ drawn	B1	3.4	∪-shaped parabola in y-z plane (axes labelled)
		The section $y = 0$, i.e. $z = f(x, 0)$ drawn	B1	3.4	∩-shaped parabola in x-z plane (axes labelled)
			[2]		
(b)		Suggestion $z = ay^2 - bx^2$	M1 *	3.3	**M** for quadratic (only) terms in both x and y
		$(-0.25 \le x \le 0.25,\ -0.3 \le y \le 0.3)$	A1	1.1	Details, including signs of coefficients (Domain not required)
		$z = 0.27$ when $x = 0, y = (\pm)\,0.3$	M1 *dep	3.4	Use of "boundary" conditions to evaluate a, b (at least one fully attempted)
		$\Rightarrow a = 3$	A1	1.1	
		$z = -0.4$ when $y = 0, x = (\pm)\,0.25 \Rightarrow b = 6.4$	A1	2.2a	
			[5]		

11.

a	$\frac{\partial z}{\partial x} = \frac{1}{2}z^{-1}.-6x$ $\quad \frac{\partial z}{\partial y} = \frac{1}{2}z^{-1}.-4y$	**M1**	1.1	Attempt at partial differentiation (at least one case) $ax \times z^{-1}$ or $by \times z^{-1}$
		A1	1.1	At least one correct (any form)
	$\frac{\partial z}{\partial x} = -\frac{3x}{z} = -11$ and $\frac{\partial z}{\partial y} = -\frac{2y}{z} = \frac{16}{3}$	**M1**	1.1	Substituting values into *their* partial derivatives to get numerical "gradients" (can be implied by a correct answer)
		A1	1.1	Both correct
	Alternative method			
	$2z\frac{\partial z}{\partial x} = -6x$ $\quad 2z\frac{\partial z}{\partial y} = -4y$	**M1**		Squaring and use of implicit differentiation (at least one case)
		A1		At least one correct (any form)
	$\frac{\partial z}{\partial x} = -\frac{3x}{z} = -11$ and $\frac{\partial z}{\partial y} = -\frac{2y}{z} = \frac{16}{3}$	**M1**		Substituting values to get numerical "gradients"
		A1		Both correct
		[4]		
b	$z - 3 = -11(x - 11) + \frac{16}{3}(y + 8)$	**M1**	1.1	Eqn. for tangent plane used Must have numerical "gradients" involved
	$\Rightarrow 33x - 16y + 3z = 500$	**A1**	1.1	**CAO** (any non-zero integer multiple)
		[2]		

12.

a	$\frac{\partial z}{\partial x} = 8x + 4y + 6$ and $\frac{\partial z}{\partial y} = 4x + 2y + 3$	**M1**	1.1	Both first partial derivatives attempted
		A1	1.1	Both correct
		[2]		
b	Considering z as a function of $2x + y$	**M1**	3.1a	
	$z = (2x + y)^2 + a(2x + y) + b$	**M1**	3.2a	a and b non-zero constants
	a=3 and b=k	**A1**	2.1	
	For S always lying above the x-y plane $z > 0$ $t = 2x + y$ so $z = t^2 + 3t + k$ (> 0) soi	**B1**	1.1	
	Discriminant of $t^2 + 3t + k$ <0	**M1**	3.1a	Or completing the square $\left(t + \frac{3}{2}\right)^2 + c$ (This implies the first two M1) and stating $c > 0$
	Discriminant=9-4k	**A1**	1.1	If completing the square, $\left(t + \frac{3}{2}\right)^2 - \frac{9}{4} + k$ This implies the first A1
	$k > \frac{9}{4}$ so least integer k is 3	**B1**	2.4	
				SC1 k=3 with no evidence
	Alternative method			
	$8x + 4y + 6 = 0$ and/or $4x + 2y + 3 = 0$	**M1**		
	$x = -\frac{3}{4} - \frac{1}{2}y$ or $y = -2x - \frac{3}{2}$	**M1**		
	Substituting either into z	**M1**		
	Correct substitution	**A1**		
	$-\frac{9}{4} + k$	**A1**		seen
	For S always lying above the x-y plane $z > 0$	**B1**		Must justify that a minimum value of z occurs when $4x + 2y + 3 = 0$ (eg through a sketch)
	$\Rightarrow k > \frac{9}{4}$ so least integer k is 3	**B1**		
		[7]		

13.

(a)	A straight line with positive gradient	**M1**	**2.2a**	i.e. with correct axes (labelled or otherwise shown or implied by an equation in *y*-*z*)
	$z = 2y + 1$ clearly shown in the *y*-*z* plane	**A1**	**1.1**	By intercepts $(-\frac{1}{2}, 0)$ and $(0, 1)$ or any other means
		[2]		
(b)	A ∪-shaped parabola	**M1**	**2.2a**	i.e. with correct axes (labelled or otherwise shown or implied by an equation in *x*-*z*)
	$z = x^2 + 2x$ clearly shown in the *x*-*z* plane	**A1**	**1.1**	By intercepts (-2,0) and (0,0) or any other means
		[2]		