Matrices – Post 2018 Mix – Mark Scheme

1.

(a)	$\lvert\mathbf{A}-\lambda\mathbf{I}\rvert=\begin{vmatrix}6-\lambda & -2 & 2\\ -2 & 3-\lambda & -1\\ 2 & -1 & 3-\lambda\end{vmatrix}=(6-\lambda)[...]-(-2)[...]+2[...]=...$	M1	1.1b
	$(6-\lambda)\left((3-\lambda)^2-1\right)+2\left(2(\lambda-3)+2\right)+2\left(2-2(3-\lambda)\right)(=0)$ $\left(\lambda^3-12\lambda^2+36\lambda-32=0\right)$	A1	1.1b
	$=(\lambda-2)\left(\lambda^2+...\lambda+...\right)$	M1	2.1
	$=(\lambda-2)\left(\lambda^2-10\lambda+16\right)=(\lambda-2)^2(\lambda-8)\Rightarrow\lambda=2$ is a repeated eigenvalue *	A1*	2.2a
	$\lambda=8$	B1	1.1b
		(5)	
(b)	$\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}\mathbf{v}=2\begin{pmatrix}x\\ y\\ z\end{pmatrix}$ or $\begin{pmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{pmatrix}\mathbf{v}=8\begin{pmatrix}x\\ y\\ z\end{pmatrix}\Rightarrow\mathbf{v}=...$	M1	1.1b
	Obtains any multiple of $\begin{pmatrix}2\\ -1\\ 1\end{pmatrix}$ for $\lambda=8$	A1	1.1b
	Obtains any (non-zero) multiple or linear combination of $\begin{pmatrix}-1\\ 0\\ 2\end{pmatrix}$ or $\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$ or $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$ for $\lambda=2$	A1	1.1b
	Obtains a **different** linear combination or (non-zero) multiple of different vector from $\begin{pmatrix}-1\\ 0\\ 2\end{pmatrix}$ or $\begin{pmatrix}1\\ 2\\ 0\end{pmatrix}$ or $\begin{pmatrix}0\\ 1\\ 1\end{pmatrix}$ for $\lambda=2$	A1	3.1a
		(4)	
(c)	Forms a matrix with their eigenvectors as columns	M1	1.2
	E.g. $\begin{pmatrix}-1 & 1 & 2\\ 0 & 2 & -1\\ 2 & 0 & 1\end{pmatrix}$	A1ft	1.1b
		(2)	

(11 marks)

2.

(a)	Sight of $\det(\mathbf{M}-\lambda\mathbf{I})=0$	**B1**	1.1a
	$\begin{vmatrix}1-\lambda & k & -2\\ 2 & -4-\lambda & 1\\ 1 & 2 & 3-\lambda\end{vmatrix}=0\Rightarrow$ $(1-\lambda)\left[(-4-\lambda)(3-\lambda)-2\right]-k\left[2(3-\lambda)-1\right]+(-2)\left[4-(-4-\lambda)\right]=0$	**M1**	1.1b
	$\Rightarrow(1-\lambda)(\lambda^2+\lambda-14)-k(5-2\lambda)-16-2\lambda=0$ $\Rightarrow\lambda^3-(2k+13)\lambda+5(k+6)=0$*	**A1***	2.1
		(3)	
(b)	(i) $\pm5(k+6)=5\Rightarrow k=\ldots$ or $(-12-2)-k(6-1)-2(4+4)=5\Rightarrow k=\ldots$	**M1**	1.1b
	$k=-7$	**A1**	2.2a
	(ii) Hence by the C-H theorem $\mathbf{M}^3+\mathbf{M}-5\mathbf{I}=\mathbf{0}$	**M1**	2.1
	Multiplying by $\mathbf{M}^{-1}$ gives $\mathbf{M}^2+\mathbf{I}-5\mathbf{M}^{-1}=\mathbf{0}\Rightarrow\mathbf{M}^{-1}=\ldots$	**M1**	3.1a
	So $\mathbf{M}^{-1}=\frac{1}{5}\left(\mathbf{M}^2+\mathbf{I}\right)$	**A1**	1.1b
	$=\frac{1}{5}\left(\begin{pmatrix}-15 & 17 & -15\\ -5 & 4 & -5\\ 8 & -9 & 9\end{pmatrix}+\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)=\ldots$	**M1**	1.1b
	$=\frac{1}{5}\begin{pmatrix}-14 & 17 & -15\\ -5 & 5 & -5\\ 8 & -9 & 10\end{pmatrix}$ or $\begin{pmatrix}-\frac{14}{5} & \frac{17}{5} & -3\\ -1 & 1 & -1\\ \frac{8}{5} & -\frac{9}{5} & 2\end{pmatrix}$	**A1**	1.1b
		(7)	

(10 marks)

3.

(a)(i)	$\begin{pmatrix} 5 & -2 & 5 \\ 0 & 3 & p \\ -6 & 6 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3-2p \\ 2 \end{pmatrix} = -1 \times \begin{pmatrix} 2 \\ 2p-3 \\ -2 \end{pmatrix}$		
	Corresponding eigenvalue is -1	**B1**	1.1b
		(1)	
a)(ii)	$2p-3=1 \Rightarrow p=...$	**M1**	1.1b
	$p=2$ *	**A1***	1.1b
		(2)	
a)(iii)	$\det\begin{pmatrix} 5-\lambda & -2 & 5 \\ 0 & 3-\lambda & p \\ -6 & 6 & -4-\lambda \end{pmatrix}=0$ $\Rightarrow (5-\lambda)((3-\lambda)(-4-\lambda)-12)-(-2)(12)+5(6(3-\lambda))=0$	**M1**	1.1b
	$\Rightarrow \lambda^3-4\lambda^2+\lambda+6=0$	**A1**	1.1b
	$\left(\Rightarrow(\lambda+1)(\lambda^2-5\lambda+6)=0 \Rightarrow (\lambda+1)(\lambda-2)(\lambda-3)=0\right)$ Eigenvalues are (-1), 2 and 3	**A1**	1.1b
	Either $\left.\begin{array}{r} 3x-2y+5z=0 \\ y+2z=0 \\ -6x+6y-6z=0 \end{array}\right\}$ or $\left.\begin{array}{r} 2x-2y+5z=0 \\ 2z=0 \\ -6x+6y-7z=0 \end{array}\right\} \Rightarrow x/y/z=...$	**M1**	2.1
	Either $k\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ (for $\lambda=2$) or $m\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (for $\lambda=3$)	**A1**	1.1b
	Both $\left.\begin{array}{r} 3x-2y+5z=0 \\ y+2z=0 \\ -6x+6y-6z=0 \end{array}\right\}$ and $\left.\begin{array}{r} 2x-2y+5z=0 \\ 2z=0 \\ -6x+6y-7z=0 \end{array}\right\} \Rightarrow x/y/z=...$	**M1**	2.1
	Both $k\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ (for $\lambda=2$) and $m\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (for $\lambda=3$)	**A1**	1.1b
		(7)	
(b)	E.g. $\mathbf{P}=\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 1 \\ -2 & -1 & 0 \end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	**B1ft**	2.2a
		(1)	

4.

(a)	$\begin{vmatrix} 1-\lambda & 0 & a \\ -3 & b-\lambda & 1 \\ 0 & 1 & a-\lambda \end{vmatrix}$ $= (1-\lambda)[(b-\lambda)(a-\lambda)-1] + a(-3)(=0)$	M1	1.1b
	$\lambda^3 - (a+b+1)\lambda^2 + (a+b+ab-1)\lambda + (3a+1-ab)(=0)$ o.e. $-\lambda^3 + (a+b+1)\lambda^2 - (a+b+ab-1)\lambda + (ab-3a-1)(=0)$ o.e.	A1	1.1b
	$\lambda^2 \Rightarrow\ a+b+1=7$ and $\lambda \Rightarrow\ a+b+ab-1=13$ Solves simultaneously e.g. $a+b=6, ab=8$ For example: leading to $a^2-6a+8=0 \Rightarrow a=\ldots$	M1	3.1a
	$a=2, b=4$	A1	1.1b
	$c=-1$	A1	2.2a
		(5)	
b)	$\mathbf{M}^3 - 7\mathbf{M}^2 + 13\mathbf{M} + \text{'their c' }\mathbf{I} = 0$	B1ft	1.1b
	$I = M^3 - 7M^2 + 13M \Rightarrow M^{-1} = M^2 - 7M + 13I$ $\Rightarrow M^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}^2 - 7\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = \ldots$ $= \begin{pmatrix} 1 & 2 & 6 \\ -15 & 17 & 0 \\ -3 & 6 & 5 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 & 2 \\ -3 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{pmatrix} = \ldots$	M1	3.1a
	$M^{-1} = \begin{pmatrix} 7 & 2 & -8 \\ 6 & 2 & -7 \\ -3 & -1 & 4 \end{pmatrix}$	A1	1.1b
		(3)	

(8 marks)

5.

(a)	$\det\begin{pmatrix}-1-\lambda & a\\ 3 & 8-\lambda\end{pmatrix}=0\Rightarrow(-1-\lambda)(8-\lambda)-3a=0\Rightarrow\ldots$	M1	1.1b
	$\lambda^2-7\lambda-3a-8=0$ o.e.	A1	1.1b
		(2)	
b)	$\mathbf{A}^2-7\mathbf{A}-3a\mathbf{I}-8\mathbf{I}=0\Rightarrow\mathbf{A}^3=7\mathbf{A}^2+(3a+8)\mathbf{A}$	M1	1.1b
	$\Rightarrow\mathbf{A}^3=7\left(7\mathbf{A}+(3a+8)\mathbf{I}\right)+(3a+8)\mathbf{A}$ Or $\mathbf{A}^3=7\begin{pmatrix}-1 & a\\ 3 & 8\end{pmatrix}^2+(8+3a)\begin{pmatrix}-1 & a\\ 3 & 8\end{pmatrix}$ $=7\begin{pmatrix}1+3a & 7a\\ 21 & 3a+64\end{pmatrix}+\begin{pmatrix}-8-3a & 8a+3a^2\\ 24+9a & 64+24a\end{pmatrix}$	M1	2.1
	$\Rightarrow\mathbf{A}^3=(3a+8+49)\mathbf{A}+7(3a+8)\mathbf{I}\Rightarrow 3a+57=1\Rightarrow a=\ldots$ Or $\begin{pmatrix}b-1 & a\\ 3 & b+8\end{pmatrix}=\begin{pmatrix}18a-1 & 3a^2+57a\\ 171+9a & 512+27a\end{pmatrix}\Rightarrow$ e.g. $3=171+9a\Rightarrow a=\ldots$ Or $\mathbf{A}^3=\begin{pmatrix}-1 & a\\ 3 & 8\end{pmatrix}+\begin{pmatrix}18a & 3a^2+56a\\ 168+9a & 504+27a\end{pmatrix}\Rightarrow$ e.g. $18a=504+27a\Rightarrow a=\ldots$	M1	1.1b
	$\Rightarrow a=-\frac{56}{3}, b=-336$	A1	1.1b
		(4)	

(6 marks)

6.

(a)	Finds the characteristic equation $(2-\lambda)^2(4-\lambda)-(4-\lambda)=0$	M1	2.1
	So $(4-\lambda)(\lambda^2-4\lambda+3)=0$ so $\lambda=4$*	A1*	2.2a
	Solves quadratic equation to give	M1	1.1b
	$\lambda=1$ and $\lambda=3$	A1	1.1b
		(4)	
(b)	Uses a correct method to find an eigenvector	M1	1.1b
	Obtains a vector parallel to one of $\begin{pmatrix}0\\0\\1\end{pmatrix}$ or $\begin{pmatrix}1\\1\\1\end{pmatrix}$ or $\begin{pmatrix}3\\-3\\1\end{pmatrix}$	A1	1.1b
	Obtains two correct vectors	A1	1.1b
	Obtains all three correct vectors	A1	1.1b
		(4)	
(c)	Uses their three vectors to form a matrix	M1	1.2
	$\begin{pmatrix}0&1&3\\0&1&-3\\1&1&1\end{pmatrix}$ **or** other correct answer with columns in a different order	A1	1.1b
		(2)	

(10 marks)