Recurrence Relations – Post 2018 Mix

1.

(i) **(a)** Solve the recurrence relation

$$X_{n+2} = 1.3X_{n+1} + 0.3X_n \text{ for } n \geqslant 0$$

given that $X_0 = 12$ and $X_1 = 1$. **[6]**

(b) Show that the sequence $\{X_n\}$ approaches a geometric sequence as n increases. **[2]**

The recurrence relation in part **(i)** models the projected annual profit for an investment company, so that X_n represents the profit (in £) at the end of year n.

(ii) **(a)** Determine the number of years taken for the projected profit to exceed one million pounds. **[2]**

(b) Compare your answer to part **(ii)(a)** with the corresponding figure given by the geometric sequence of part **(i)(b)**. **[2]**

2.

The members of the family of the sequences $\{u_n\}$ satisfy the recurrence relation

$$u_{n+1} = 10u_n - u_{n-1} \text{ for } n \geqslant 1. \qquad (*)$$

(i) Determine the general solution of **(*)**. **[3]**

(ii) The sequences $\{a_n\}$ and $\{b_n\}$ are members of this family of sequences, corresponding to the initial terms $a_0 = 1$, $a_1 = 5$ and $b_0 = 0$, $b_1 = 2$ respectively.

(a) Find the next two terms of each sequence. **[1]**

(b) Prove that, for all non-negative integers n, $(a_n)^2 - 6(b_n)^2 = 1$. **[8]**

(c) Determine $\lim_{n \to \infty} \left(\frac{a_n}{b_n} \right)$. **[2]**

3.

A sequence $\{u_n\}$ is given by $u_{n+1} = 4u_n + 1$ for $n \geqslant 1$ and $u_1 = 3$. (*)

(a) Find the values of u_2, u_3 and u_4. **[1]**

(b) Solve the recurrence system (*). **[5]**

4.

(i) Solve the recurrence relation $u_{n+2} = 4u_{n+1} - 4u_n$ for $n \geq 0$, given that $u_0 = 1$ and $u_1 = 1$. **[4]**

(ii) Show that each term of the sequence $\{u_n\}$ is an integer. **[2]**

5.

In order to rescue them from extinction, a particular species of ground-nesting birds is introduced into a nature reserve. The number of breeding pairs of these birds in the nature reserve, t years after their introduction, is an integer denoted by N_t. The initial number of breeding pairs is given by N_0.

An initial discrete population model is proposed for N_t.

$$\text{Model I: } N_{t+1} = \tfrac{6}{5} N_t \left(1 - \tfrac{1}{900} N_t\right)$$

(i) (a) For Model I, show that the steady state values of the number of breeding pairs are 0 and 150. **[3]**

(b) Show that $N_{t+1} - N_t < 150 - N_t$ when N_t lies between 0 and 150. **[3]**

(c) Hence find the long-term behaviour of the number of breeding pairs of this species of birds in the nature reserve predicted by Model I when $N_0 \in (0, 150)$. **[2]**

6.

The sequence $\{u_n\}$ is defined by $u_0 = 2$, $u_1 = 5$ and $u_n = \dfrac{1+u_{n-1}}{u_{n-2}}$ for $n \geqslant 2$.

Prove that the sequence is periodic with period 5. **[4]**

7.

(a) Solve the second-order recurrence relation $T_{n+2} + 2T_n = -87$ given that $T_0 = -27$ and $T_1 = 27$. **[8]**

(b) Determine the value of T_{20}. **[2]**

8.

Solve the second-order recurrence system $H_{n+2} = 5H_{n+1} - 4H_n$ with $H_0 = 3,\ H_1 = 7$ for $n \geqslant 0$. **[5]**

9.

For each value of k the sequence of real numbers $\{u_n\}$ is given by $u_1 = 2$ and $u_{n+1} = \dfrac{k}{6+u_n}$.

For each of the following cases, either determine a value of k or prove that one does not exist.

(a) $\{u_n\}$ is constant. **[2]**

(b) $\{u_n\}$ is periodic, with period 2. **[3]**

(c) $\{u_n\}$ is periodic, with period 4. **[5]**

10.

In a national park, the number of adults of a given species is carefully monitored and controlled. The number of adults, n months after the start of this project, is A_n. Initially, there are 1000 adults. It is predicted that this number will have declined to 960 after one month.

The first model for the number of adults is that, from one month to the next, a fixed proportion of adults is lost. In order to maintain a fixed number of adults, the park managers "top up" the numbers by adding a constant number of adults from other parks at the end of each month.

(a) Use this model to express the number of adults as a first-order recurrence system. **[1]**

Instead, it is found that, the proportion of adults lost each month is double the predicted amount, with no change being made to the constant number of adults added each month.

(b) **(i)** Show that the revised recurrence system for A_n is $A_0 = 1000$, $A_{n+1} = 0.92A_n + 40$. **[1]**

(ii) Solve this revised recurrence system. **[4]**

(iii) Describe the long-term behaviour of the sequence $\{A_n\}$ in this case. **[1]**

A more refined model for the number of adults uses the second-order recurrence system $A_{n+1} = 0.9A_n - 0.1A_{n-1} + 50$, for $n \geqslant 1$, with $A_0 = 1000$ and $A_1 = 920$.

(c) **(i)** Determine the long-term behaviour of the sequence $\{A_n\}$ for this more refined model. **[4]**

11.

The number of visits to a website, in any particular month, is modelled as the number of visits received in the previous month plus k times the number of visits received in the month before that, where k is a positive constant.

Given that V_n is the number of visits to the website in month n,

(a) write down a general recurrence relation for V_{n+2} in terms of V_{n+1}, V_n and k.

(1)

For a particular website you are given that

- $k = 0.24$
- In month 1, there were 65 visits to the website.
- In month 2, there were 71 visits to the website.

(b) Show that

$$V_n = 50(1.2)^n - 25(-0.2)^n$$

(5)

This model predicts that the number of visits to this website will exceed one million for the first time in month N.

(c) Find the value of N.

(2)

12.

Solve the recurrence system

$$u_1 = 1 \qquad u_2 = 4$$
$$9u_{n+2} - 12u_{n+1} + 4u_n = 3n$$

(9)

13.

(a) Determine the general solution of the recurrence relation

$$u_n = 2u_{n-1} - u_{n-2} + 2^n \qquad n \geqslant 2$$

(4)

(b) Hence solve this recurrence relation given that $u_0 = 2u_1$ and $u_4 = 3u_2$

(2)

14.

Determine a closed form for the recurrence relation

$$u_0 = 1 \qquad u_1 = 4$$

$$u_{n+2} = 2u_{n+1} - \frac{4}{3}u_n + n \qquad n \geqslant 0$$

(7)

15.

A staircase has n steps. A tourist moves from the bottom (step zero) to the top (step n). At each move up the staircase she can go up either one step or two steps, and her overall climb up the staircase is a combination of such moves.

If u_n is the number of ways that the tourist can climb up a staircase with n steps,

(a) explain why u_n satisfies the recurrence relation

$$u_n = u_{n-1} + u_{n-2}, \text{ with } u_1 = 1 \text{ and } u_2 = 2$$

(3)

(b) Find the number of ways in which she can climb up a staircase when there are eight steps.

(1)

A staircase at a certain tourist attraction has 400 steps.

(c) Show that the number of ways in which she could climb up to the top of this staircase is given by

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$$

(5)