Recurrence Relations – Post 2018 Mix – Mark Scheme

1.

(i)	(a)	Characteristic Equation is $\lambda^2 - 1.3\lambda - 0.3 = 0$	M1	1.1a	
		$\Rightarrow \lambda = 1.5, -0.2$	A1	1.1	BC
		$\Rightarrow$ General Solution is $X_n = A \times 1.5^n + B \times (-0.2)^n$	B1	1.2	FT their λs
		Use of $X_0 = 12$ and $X_1 = 1$ to obtain equations in A, B: i.e. $12 = A + B$ & $1 = 1.5A - 0.2B$	M1	1.1a	
		Solving $\Rightarrow A = 2,\ B = 10$	M1	1.1	Eqns. solved simultaneously (e.g. **BC**)
		Solution is $X_n = 2 \times 1.5^n + 10 \times (-0.2)^n$	A1	2.5	**cao** brackets required
			[6]		
	(b)	For large n, $(-0.2)^n \to 0$	B1	3.1b	
		so $X_n \to 2 \times 1.5^n$ which is of the form $a\,r^n$, hence a GP	E1	2.2a	**OR** $3 \times 1.5^{n-1}$, of the form $a\,r^{n-1}$, hence a GP [MUST have some explanation that this is a GP]
			[2]		
(ii)	(a)	Tabulating the given sequence (or using calculator equation solver)	M1	3.1a	**BC** or manual calculation
		$X_{32} \approx 862\,880 < 1\,000\,000$ and $X_{33} \approx 1\,294\,320 > 1\,000\,000$ so $n = 33$	A1	3.2a	Properly justified
			[2]		
	(b)	$X_n = 2 \times 1.5^n > 1\,000\,000$	M1	2.1	Attempt at solving (logs not essential)
		$\Rightarrow n > \dfrac{\log 500000}{\log 1.5} = 32.36\ldots$ so $n = 33$	A1	1.1	Properly justified
			[2]		

2.

(i)		Aux. Eqn. is $\lambda^2 - 10\lambda + 1 = 0 \Rightarrow \lambda = 5 \pm \sqrt{24}$ or $5 \pm 2\sqrt{6}$	M1 A1	1.1a 1.1	
		Gen. Soln. is then $u_n = A\left(5 + 2\sqrt{6}\right)^n + B\left(5 - 2\sqrt{6}\right)^n$	B1	1.1	**FT** their two values of λ
			[3]		
(ii)	(a)	$\{a_n\} = \{1, 5, \mathbf{49}, \mathbf{485}, \ldots\}$ and $\{b_n\} = \{0, 2, \mathbf{20}, \mathbf{198}, \ldots\}$	B1	1.1	All highlighted terms required
			[1]		
	(b)	$a_0 = 1,\ a_1 = 5 \Rightarrow 1 = A + B$ and $5 = 5(A + B) + (A - B)\,2\sqrt{6}$	M1	1.1a	
		$\Rightarrow A = B = \frac{1}{2}$ i.e. $a_n = \frac{1}{2}\left(5 + 2\sqrt{6}\right)^n + \frac{1}{2}\left(5 - 2\sqrt{6}\right)^n$	A1	1.1	Sim. eqns. may be solved by GC, for instance
		$b_0 = 0,\ b_1 = 2 \Rightarrow 0 = A + B$ and $2 = 5(A + B) + (A - B)\,2\sqrt{6}$	M1	1.1a	
		$\Rightarrow A = -B = \dfrac{1}{2\sqrt{6}}$ i.e. $b_n = \dfrac{1}{2\sqrt{6}}\left(5 + 2\sqrt{6}\right)^n - \dfrac{1}{2\sqrt{6}}\left(5 - 2\sqrt{6}\right)^n$	A1	1.1	Sim. eqns. may be solved by GC, for instance
		Either Calling $\alpha = 5 + 2\sqrt{6}$ and $\beta = 5 - 2\sqrt{6}$ and substituting both in	M1	3.1a	The shorthand is helpful but not necessary
		$\left(a_n\right)^2 - 6\left(b_n\right)^2 = \frac{1}{4}\left\{\alpha^{2n} + 2(\alpha\beta)^n + \beta^{2n}\right\} - 6 \times \frac{1}{24}\left\{\alpha^{2n} - 2(\alpha\beta)^n + \beta^{2n}\right\}$	A1	2.1	
		$= (\alpha\beta)^n = 1$, by the "*difference of two squares*" (or product of roots)	A1 A1	2.5 2.2a	$\alpha\beta = 1$ must at least be stated
		Or $\left(a_n\right)^2 - 6\left(b_n\right)^2 = \left(a_n - \sqrt{6}b_n\right)\left(a_n + \sqrt{6}b_n\right)$	M1		
		$= \frac{1}{2}\left\{\alpha^n + \beta^n - \alpha^n + \beta^n\right\} \cdot \frac{1}{2}\left\{\alpha^n + \beta^n + \alpha^n - \beta^n\right\}$	M1 A1		
		$= (\alpha\beta)^n = 1$ as above	A1		

(ii)	(c)	**Either** For large n, $\left(a_n\right)^2 \approx 6\left(b_n\right)^2$	M1	2.1	
		$\Rightarrow \dfrac{a_n}{b_n} \to \sqrt{6}$	A1	2.1	
		Or $\lvert\beta\rvert < 1 \Rightarrow a_n \to \frac{1}{2}\alpha^n$ and $b_n \to \frac{1}{2\sqrt{6}}\alpha^n$	M1		
		so that $\dfrac{a_n}{b_n} \to \dfrac{\frac{1}{2}}{\frac{1}{2\sqrt{6}}} = \sqrt{6}$	A1		
			[2]		

3.

(i)	$u_2 = 13,\ u_3 = 53,\ u_4 = 213$	B1 [1]	1.1	All three
(ii)	(Auxiliary Equation is $m - 4 = 0$)			
	so that Complementary Solution (CS) is $u_n = A(4^n)$	B1	1.2	
	Particular Solution (PS) is $u_n = a$ with $a - 4a = 1$ i.e. PS is $u_n = -\frac{1}{3}$	B1	1.1	
	(General Solution is $u_n = A(4^n) - \frac{1}{3}$)	B1FT	1.1	**FT** provided CS has 1 arbitrary constant, PS none **soi**
	Using $n = 1,\ u_1 = 3,$ to find the value of A	M1	1.1a	
	$u_n = \frac{5}{6}(4^n) - \frac{1}{3}$ or equivalent	A1 [5]	1.1	or $A = \frac{5}{6}$

4.

(i)	Characteristic Equation is $\lambda^2 - 4\lambda + 4 = 0$	M1	1.1a		
	$\Rightarrow \lambda = 2$ (twice)				
	and General Solution is $u_n = (A + Bn) \times 2^n$	A1	1.1		
	$u_0 = 1$ and $u_1 = 1 \Rightarrow 1 = A$ and $1 = 2(A + B)$	M1	1.1	Substitution of first two known terms	
	$\Rightarrow A = 1, B = -\frac{1}{2}$ and $u_n = \left(1 - \frac{1}{2}n\right) \times 2^n$	A1 FT [4]	1.1	FT correctly solved sim. eqns.	
(ii)	So $u_n = (2 - n) \times 2^{n-1}$	M1	3.1a		Do not penalise lack of checking for $u_0 = 1$ since this is given
	and both parts of the product are integers so u_n is an integer	E1 [2]	2.4		

5.

(i)	(a)	$N_{t+1} - N_t = \frac{1}{5}N_t - \frac{1}{750}N_t^2$ or $\frac{1}{750}N_t(150 - N_t)$	B1	3.4	
		Let M be a steady state value, then	M1	2.1	
		$\frac{1}{750}M(150 - M) = 0$			
		$M = 0$ or 150	A1 [3]	1.1	
(i)	(b)	$N_{t+1} - N_t = \frac{1}{750}N_t(150 - N_t)$	M1	1.1	soi
		$N_t \in (0,150) \Rightarrow \frac{1}{750}N_t \in (0, \frac{1}{5})$	M1	2.1	
		Therefore, since $150 - N_t > 0$			
		We have that $N_{t+1} - N_t < 150 - N_t$	E1 [3]	2.2a	
(i)	(c)	N_t increases to approach 150 in the long term, without oscillation	B1 B1 [2]	3.4 3.4	For "150" For "without oscillation" oe

6.

$u_2 = 3$	1.1a	M1	Use of the given r.r. (with at least u_2 correct)
$u_3 = 0.8$	1.1	M1	Repeated use of the given r.r. (with at least u_3 correct)
$u_4 = 0.6$, $u_5 = 2$ and $u_6 = 5$	1.1	A1	u_4, u_5 & u_6 correct
Explanation that $u_0 = u_5$ and $u_1 = u_6 \Rightarrow$ periodicity, period 5	2.4	E1 [4]	(since a second-order r.r.)

7.

(a)	PS is $T_n = -29$	1.1	B1	Particular Solution
	CS from $\lambda^2 + 2 = 0$, $\lambda = \pm i\sqrt{2}$, is $T_n = A\left(i\sqrt{2}\right)^n + B\left(-i\sqrt{2}\right)^n$	1.1a 1.1	M1 A1	Complementary Solution attempted from auxiliary equation; correct
	General Solution is $T_n = A\left(i\sqrt{2}\right)^n + B\left(-i\sqrt{2}\right)^n - 29$	1.2	B1	**FT** GS = CS (with 2 arbitrary constants) + PS (with none)
	Use of $T_0 = -27$ and $T_1 = 27$ to create simultaneous equations in A, B	1.1a	M1	
	Two correct equations: $A + B = 2$ and $A - B = -28i\sqrt{2}$	1.1	A1	Or unsimplified equivalents
	Solving attempt at two equations: $A = 1 - 14i\sqrt{2}$ and $B = 1 + 14i\sqrt{2}$	1.1a	M1	
	i.e. $T_n = \left(1 - 14i\sqrt{2}\right)\left(i\sqrt{2}\right)^n + \left(1 + 14i\sqrt{2}\right)\left(-i\sqrt{2}\right)^n - 29$	1.1	A1	cao
	Alt. version			
	PS is $T_n = -29$		B1	Particular Solution
	CS from $\lambda^2 + 2 = 0$, $\lambda = \pm i\sqrt{2}$, is $T_n = (\sqrt{2})^n\left(C\cos\frac{n\pi}{2} + D\sin\frac{n\pi}{2}\right)$		M1 A1	Complementary Solution attempted from auxiliary equation; correct
	General Solution is $(\sqrt{2})^n\left(C\cos\frac{n\pi}{2} + D\sin\frac{n\pi}{2}\right) - 29$		B1	**FT** GS = CS (with 2 arbitrary constants) + PS (with none)
	Use of $T_0 = -27$ and $T_1 = 27$ to create simultaneous equations in C, D		M1	
	Two correct equations: $C - 29 = -27$ and $\sqrt{2}\,D - 29 = 27$		A1	Unsimplified
	Solving attempt at two equations: $C = 2$ and $D = 28\sqrt{2}$		A1	
	i.e. $T_n = (\sqrt{2})^n\left(2\cos\frac{n\pi}{2} + 28\sqrt{2}\sin\frac{n\pi}{2}\right) - 29$		A1	cao
			[8]	
(b)	Substituting $n = 20 \Rightarrow$ $T_{20} = \left(1 - 14i\sqrt{2}\right) \times 1024 + \left(1 + 14i\sqrt{2}\right) \times 1024 - 29$	2.1	M1	**Or** by applying the given recurrence relation (e.g. **BC**)
	$= 2048 - 29 = 2019$	1.1	A1 [2]	

8.

Aux. Eqn. is $m^2 - 5m + 4 = 0 \Rightarrow m = 1, 4$	M1	1.1	
$\Rightarrow$ Gen. Soln. is $H_n = A + B \times 4^n$	A1	1.1	
$H_0 = 3 \Rightarrow 3 = A + B$ and $H_1 = 7 \Rightarrow 7 = A + 4B$	M1	1.1	Use of initial terms
$\Rightarrow A = \frac{5}{3}$, $B = \frac{4}{3}$	M1	1.1	Solving simultaneous eqns. in A and B
giving $H_n = \frac{1}{3}\left(5 + 4^{n+1}\right)$	A1	1.1	
	[5]		

9.

(a)	For constant sequence, set $2 = \frac{k}{6+2}$ (from $u_2 = u_1$)	M1	1.1a	
	$\Rightarrow k = 16$	A1 [2]	1.1	
(b)	$u_3 = \frac{k}{6 + \frac{k}{8}} = \frac{8k}{48 + k}$ equated to $u_1 = 2$	M1	3.1a	u_3 correct (simplified) and set = 2
	Solving for $k \Rightarrow 96 + 2k = 8k \Rightarrow k = 16$	A1	1.1	
	But this is the condition for $\{u_n\}$ constant, so the sequence is never periodic with period 2	A1	2.3	Correct conclusion stated with supporting reason
		[3]		
(c)	$u_4 = \frac{k}{6 + \frac{8k}{48+k}} = \frac{k(k+48)}{288 + 14k}$, $u_5 = \frac{k}{6 + \frac{k(k+48)}{288+14k}} = \frac{k(14k + 288)}{k^2 + 132k + 1728}$	M1 A1	1.1 1.1	u_4 and u_5 attempted in terms of k At least u_4 correct (simplified)
	For $\{u_n\}$ periodic, period 4: $u_5 = u_1$ $\Rightarrow 14k^2 + 288k = 2k^2 + 264k + 3456$	M1	3.1a	u_5 must have been worked out and an algebraic expression equated to 2
	$12(k^2 + 2k - 288) = 0 \Rightarrow (k - 16)(k + 18) = 0$	M1	1.1	Solving a quadratic eqn. in k
	$k = -18$	A1 [5]	2.2a	Correct (single) answer only

10.

a		$A_{n+1} = 0.96A_n + 40$ and $A_0 = 1000$	B1 [1]	3.3	or $A_n = 0.96A_{n-1} + 40$ and $A_0 = 1000$
b	i	80, i.e. 8% of adults lost linked to the 0.92 (noting that the 1000 and 40 remain the same as before) $A_{n+1} = 0.92A_n + 40$ and $A_0 = 1000$	B1 [1]	2.1	**AG**
	ii	PS from $A_{n+1} = A_n = a$ is $A_n = 500$	B1	1.1	
		CS is $A_n = C \times 0.92^n$	B1	1.1	
		Soln. from GS: $A_n = C \times 0.92^n + 500$ with $A_0 = 1000$	M1	1.1	
		$A_n = 500 \times 0.92^n + 500$	A1 [4]	1.1	
	iii	A_n decreases, stabilising at a constant 500 i.e. $A_n \to 500$	B1 [1]	3.4	Not required to say "monotonic decreasing", since only *long-term* behaviour asked-for
c	i	PS is $A_n = 250$	B1	1.1	May be implied by following answer
		General solution is of the form $A_n = D \times \alpha^n + E \times \beta^n +$ "250" where α and β are positive and less than 1	M1	3.1a	Accept statement that α and β are both positive and less than 1
		Hence both α^n and $\beta^n \to 0$	A1	2.4	If calculated, α and β must be correct. Note that $\alpha = \frac{1}{2}(0.9 + \sqrt{0.41}) = 0.77$ and $\beta = \frac{1}{2}(0.9 - \sqrt{0.41}) = 0.13$ ($D = 375 + \frac{665}{2\sqrt{0.41}} \approx 894.27776$ and $E = 375 - \frac{665}{2\sqrt{0.41}} \approx -144.27776$)
		$A_n \to$ '250'	A1 [4]	1.1	**FT** from PS. **SC1** $X = 0.9X - 0.1X + 50$, so $X = 250$. **B1M1A0A1** is possible

11.

3(a)	$V_{n+2} = V_{n+1} + kV_n$	B1	3.3
		(1)	
(b)	$\lambda^2 - \lambda - 0.24 = 0 \Rightarrow \lambda = ...(1.2, -0.2)$	M1	1.1b
	$V_n = a(1.2)^n + b(-0.2)^n$	A1	2.2a
	$65 = a(1.2)^1 + b(-0.2)^1$ **and** $71 = a(1.2)^2 + b(-0.2)^2$	B1ft	3.4
	E.g. $\left.\begin{matrix} 78 = 1.44a - 0.24b \\ 71 = 1.44a + 0.04b \end{matrix}\right\} \Rightarrow 7 = -0.28b \Rightarrow b = ...$	M1	2.1
	$a = 50, b = -25 \Rightarrow V_n = 50(1.2)^n - 25(-0.2)^n$ *	A1*	1.1b
		(5)	
(c)	$50(1.2)^N > 10^6 \Rightarrow N = ...$	M1	3.1b
	$\Rightarrow N = 55$ i.e. month 55	A1	3.2a
		(2)	

(8 marks)

12.

Auxiliary equation is $9r^2 - 12r + 4 = 0$, so $r = ...$	**M1**	1.1b
$(3r-2)^2 = 0 \Rightarrow r = \frac{2}{3}$ is repeated root.	**A1**	1.1b
Complementary function is $x_n = (A+Bn)\left(\frac{2}{3}\right)^n$ or $A\left(\frac{2}{3}\right)^n + Bn\left(\frac{2}{3}\right)^n$	**M1**	2.2a
Try particular solution $y_n = an + b \Rightarrow 9(a(n+2)+b) - 12(a(n+1)+b) + 4(an+b) = 3n$	**M1**	2.1
$\Rightarrow an + 6a + b = 3n \Rightarrow a = ..., b = ...$	**dM1**	1.1b
$a = 3, b = -18$	**A1**	1.1b
General solution is $u_n = x_n + y_n = (A+Bn)\left(\frac{2}{3}\right)^n + 3n - 18$	**B1ft**	2.2a
$\left.\begin{array}{l} u_1 = 1 \Rightarrow 1 = \left(\frac{2}{3}\right)(A+B) - 15 \\ u_2 = 4 \Rightarrow 4 = \left(\frac{4}{9}\right)(A+2B) - 12 \end{array}\right\} A = ..., B = ...$	**M1**	2.1
$u_n = 12(n+1)\left(\frac{2}{3}\right)^n + 3n - 18$ oe	**A1**	1.1b
	(9)	

(9 marks)

13.

(a)	$l^{2} - 2l + 1 = 0$ Þ $l = \ldots\{\lambda = 1\}$	M1	1.1b
	$u_n = A + Bn$	A1	2.2a
	$u_n = \mu(2)^n \Rightarrow \mu(2)^n = 2\mu(2)^{n-1} - \mu(2)^{n-2} + (2)^n$ $\Rightarrow 4\mu(2)^{n-2} = 4\mu(2)^{n-2} - \mu(2)^{n-2} + 4(2)^{n-2}$ $\Rightarrow \mu = \ldots\{4\}$	M1	1.1b
	$u_n = A + Bn + 4(2)^n$ or $u_n = A + Bn + (2)^{n+2}$	A1	1.1b
		(4)	
(b)	Uses the information to find the values of the constants For example $u_0 = 2u_1 \Rightarrow A + 4 = 2(A + B + 4(2)) \Rightarrow \ldots\{A + 2B = -12\}$ $u_4 = 3u_2 \Rightarrow A + 4B + \text{'4'}(2)^4 = 3(A + 2B + 4(2)^2) \Rightarrow \ldots\{2A + 2B = 16\}$ Solves simultaneous equations to find values for *A* and *B*. **Alternatively** $u_2 = 2u_1 - u_0 + 2^2 = u_0 - u_0 + 4 = 4$ leading to $u_4 = 3u_2 = 3 \times 4 = 12$ $u_3 = 2u_2 - u_1 + 2^3 = 2 \times 4 - \left(\frac{A+4}{2}\right) + 8 = 16 - \left(\frac{A+4}{2}\right)$ $u_4 = 2u_3 - u_2 + 2^4 = 2\left(16 - \left(\frac{A+4}{2}\right)\right) - 4 + 16 = 12$ Leading $A = \ldots$ $u_2 = \text{'28'} + 2B + 4(2^2) = 4 \Rightarrow B = \ldots$	M1	3.1a
	$u_n = 28 - 20n + 4(2)^n$ or $u_n = 28 - 20n + (2)^{n+2}$	A1	1.1b
		(2)	

(6 marks)

14.

$\lambda^2 = 2\lambda - \frac{4}{3} \Rightarrow 3\lambda^2 - 6\lambda + 4 = 0 \Rightarrow \lambda = ...$	M1	1.1b
$\lambda = \frac{6 \pm \sqrt{36-48}}{6} = \frac{3 \pm i\sqrt{3}}{3} = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} \pm \frac{1}{2}i\right) = \frac{2\sqrt{3}}{3}e^{\pm i\frac{\pi}{6}}$	A1	1.1b
CF is $u_n = A\left(\frac{3+i\sqrt{3}}{3}\right)^n + B\left(\frac{3-i\sqrt{3}}{3}\right)^n$ o.e. or $\left(\frac{2\sqrt{3}}{3}\right)^n\left(P\cos\left(\frac{\pi n}{6}\right) + Q\sin\left(\frac{\pi n}{6}\right)\right)$	A1ft	2.2a
$u_n = kn + l \Rightarrow k(n+2) + l = 2k(n+1) + 2l - \frac{4}{3}(kn+l) + n$ $\Rightarrow \left(1 - \frac{1}{3}k\right)n - \frac{1}{3}l = 0 \Rightarrow k = ..., l = ...\ \ (k = 3, l = 0)$	M1	1.1b
$u_n = A\left(\frac{3+i\sqrt{3}}{3}\right)^n + B\left(\frac{3-i\sqrt{3}}{3}\right)^n + 3n$ o.e. or $u_n = \left(\frac{2\sqrt{3}}{3}\right)^n\left(P\cos\left(\frac{\pi n}{6}\right) + Q\sin\left(\frac{\pi n}{6}\right)\right) + 3n$	A1	1.1b
$\left.\begin{array}{l} u_0 = 1 \Rightarrow A + B = 1 \\ u_1 = 4 \Rightarrow A + B + (A-B)\frac{i\sqrt{3}}{3} = 1 \end{array}\right\} \Rightarrow (A-B)\frac{i\sqrt{3}}{3} = 0 \Rightarrow A = ..., B = ...$ $\left.\begin{array}{l} u_0 = 1 \Rightarrow P + 0Q = 1 \\ u_1 = 4 \Rightarrow \frac{2}{\sqrt{3}}\left(P\frac{\sqrt{3}}{2} + \frac{Q}{2}\right) = 1 \end{array}\right\} \Rightarrow P = ... \Rightarrow Q = ...$ Solves simultaneous equations to find values for A and B or P and Q	M1	3.1a
$\left(A = B = \frac{1}{2} \text{ or } P = 1, Q = 0\right)$ $\Rightarrow u_n = \frac{1}{2}\left(\frac{3+i\sqrt{3}}{3}\right)^n + \frac{1}{2}\left(\frac{3-i\sqrt{3}}{3}\right)^n + 3n$ o.e. Or $u_n = \left(\frac{2\sqrt{3}}{3}\right)^n \cos\left(\frac{\pi n}{6}\right) + 3n$	A1	1.1b
	(7)	

(7 marks)

15.

(a)	$u_1 = 1$ as there is only one way to go up one step	B1	2.4
	$u_2 = 2$ as there are two ways: one step then one step or two steps	B1	2.4
	If first move is one step then can climb the other $(n-1)$ steps in u_{n-1} ways. If first move is two steps can climb the other $(n-2)$ steps in u_{n-2} ways so $u_n = u_{n-1} + u_{n-2}$	B1	2.4
		(3)	
(b)	Sequence begins 1, 2, 3, 5, 8, 13, 21, 34,... so 34 ways of climbing 8 steps	B1	1.1b
		(1)	
(c)	To find general term use $u_n = u_{n-1} + u_{n-2}$ gives $\lambda^2 = \lambda + 1$	M1	2.1
	This has roots $\frac{1 \pm \sqrt{5}}{2}$	A1	1.1b
	So general form is $A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$	M1	2.2a
	Uses initial conditions to find A and B reaching two equations in A and B	M1	1.1b
	Obtains $A = \left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)$ and $B = -\left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)$ and so when $n = 400$ obtains $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{401} - \left(\frac{1-\sqrt{5}}{2}\right)^{401}\right]$ *	A1*	1.1b
		(5)	

(9 marks)