Year 12 Further Pure Revision (no calculus)

Summer Term 2024

Contents

1 Complex Numbers

1.1 Level 1

1. Simplify each of the following:
   (a) $(1 + i)(2 - 3i)$
   (b) $(x + 2yi)(2x + yi)$
   (c) $(p + 2qi)(p - 2qi)$
   (d) i^n for $n = 2, 3, 4, 5....$
   (e) $\frac{1}{2+3i}$
   (f) $\frac{3i-2}{1+2i}$

2. Find the modulus and argument of the following complex numbers.
   (a) $2 + i$
   (b) $-1 + 2i$
   (c) $3i$
   (d) $\cos\theta - i\sin\theta$
   (e) $\sin\theta + i\cos\theta$

3. Solve for z
   (a) $3 + 4i + z\,(2 + i) = 1$
   (b) $\dfrac{2+i}{iz} = 5 + i$
   (c) $z^2 - 4z + 29 = 0$
   (d) $z^2 + iz + 6 = 0$

   In part c all the coefficients were real numbers and in part d this was not the case. How is this reflected in your answers?

4. Given that $z = 3+4i$ and $w = 12+5i$ find the modulus and argument of the following. Give arguments as decimals rounded to 1 d.p. where $-180^o < arg(z) < 180^o$
   (a) z
   (b) w
   (c) $\frac{1}{z}$
   (d) $\frac{1}{w}$
   (e) zw
   (f) z^*
   (g) w^*
   (h) $(zw)^*$
   (i) w^2
   (j) z^{10} (don't calculate z^{10}!)

1.2 Level 2

5. For $z \in \mathbb{C}$, $z = a + bi$ for $a, b \in \mathbb{R}$, find z^2 and $\frac{1}{z}$.
   Hence verify that $|z^2| = |z|^2$ and $|\frac{1}{z}| = \frac{1}{|z|}$

6. Prove that $zz^* = |z|^2$

7. (a) The equation $x^4 - 4x^3 + 3x^2 + 2x - 6 = 0$ has a root $z = 1 - i$. Find the other three roots.
   (b) Given that $1, w_1$ and w_2 are the roots of the equation $z^3 = 1$, express w_1 and w_2 in the form $a + bi$ and show that
      i. $1 + w_1 + w_2 = 0$
      ii. $\frac{1}{w_1} = w_2$

8. The complex number z is such that $|z| = 1$ prove that:
   (a)
   $$\frac{1}{z} = z^*$$
   (b)
   $$-\frac{\pi}{2} < \text{arg(z+1)} < \frac{\pi}{2}$$

9. Simplify
   $$\left(1 + \sqrt{3}i\right)^4 - \left(1 - \sqrt{3}i\right)^4$$

10. Find the modulus and argument of the complex number $\omega = 1 + i$. Hence find an expression for ω^{101}.

2 Polynomials

2.1 Factor and Remainder Theorem

2.1.1 Level 1

11. The polynomial $x^3 + bx^2 + cx + d$ has roots $x = 2, x = 3$ and $x = 4$. Find a, b and c.

12. Find the quotient and remainder when $7x^3 - x^2 + 3x - 4$ is divided by $x - 1$.

2.1.2 Level 2

13. By using the identity $x^4 + 1 \equiv x^4 + 2x^2 + 1 - 2x^2$, solve the equation
    $$x^4 + 1 = 0$$
    Mark these roots on the Argand Diagram.

14. The equations $2x^3 - x^2 + 4kx + k = 0$ and $x^2 - x - 2 = 0$ have a common root. Find the possible values of k.

15. Prove that if $x^3 + px + q = 0$ has a repeated root if $4p^3 + 27q^2 = 0$

16. Solve the equation
$$x^4 + x^3 - 16x^2 - 4x + 48 = 0$$
Given that the product of two of the roots is 6.

2.2 Algebra of roots

2.2.1 Level 1

17. The equation $2x^2 + 4x - 5 = 0$ has roots α, β. Find the equations with roots
    (a) $\alpha + 1, \beta + 1$
    (b) $3\alpha, 3\beta$
    (c) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

18. The equation $2x^3 + 3x^2 - 13x - 7 = 0$ has roots α, β and γ. Find the equations with roots
    (a) $\alpha + 1, \beta + 1, \gamma + 1$
    (b) $\alpha - 2, \beta - 2, \gamma - 2$
    (c) $2\alpha, 2\beta, 2\gamma$
    (d) $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

19. Solve the following equations, using the hints.
    (a) $x^3 + 4x^2 - 9x - 36 = 0$, given that two of the roots sum to 0.
    (b) $x^3 + x^2 - 22x - 40 = 0$, given that one of the roots is twice another.
    (c) $16x^3 + 8x^2 - 39x + 18 = 0$, given that the equation has a double root.
    (d) $8x^3 - 24x^2 + 20x - 3 = 0$, given that one root is the sum of the other two.
    (e) $x^3 - 21x^2 + 126x - 216 = 0$, given that the roots form a geometric sequence.
    (f) $x^3 - 15x^2 + 71x - 105 = 0$, given that the roots form an arithmetic sequence.

2.2.2 Level 2

20. A rectangle with length x cm and width y cm has perimeter 12 cm and area 5 cm^2.
    (a) Write down the values of
        i. $x + y$
        ii. xy
    (b) Write down a quadratic equation satisfied by x and y,
    (c) *Without solving the quadratic* find the length of the diagonal of the rectangle.

21. A cuboid has perimeter 100 cm, surface area 80 cm^2 and volume 10 cm^3.
    (a) Write down a cubic equation satisfied by the lengths of the cuboid.
    (b) Find the length of the space diagonal of the cuboid.

22. If α, β and γ are roots of $x^3 - 2x^2 + x - 6 = 0$ find the values of:
    (a) $\alpha^2 + \beta^2 + \gamma^2$
    (b) $\alpha^3 + \beta^3 + \gamma^3$

23. If α, β and γ are roots of $x^3 - ax^2 + bx - c = 0$. Prove:
    (a) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \alpha\gamma + \alpha\beta)$
    (b) $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \beta\gamma + \alpha\gamma + \alpha\beta)$

24. Establish the identity
$$\frac{x^2 - a^2}{(a-b)(a-c)} + \frac{x^2 - b^2}{(b-c)(b-a)} + \frac{x^2 - c^2}{(c-a)(c-b)} + 1 \equiv 0$$

25. Simplify
$$(a + b - c)^2 + (b + c - a)^2 + (c + a - b)^2$$

3 Matrices

3.1 Level 1

26. You are given that:
    $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$
    Calculate, where possible:
    (a) $3A$
    (b) AB
    (c) AC
    (d) CA
    (e) $A + C$
    (f) $A + B$

27. Find where possible the inverses of the following matrices
    (a) $\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$
    (b) $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$
    (c) $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

(d) $\begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix}$

(e) $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$

28. The matrix, $\mathbf{M} = \begin{pmatrix} 7 & -3 \\ -4 & 6 \end{pmatrix}$ defines a transformation in the (x,y) plane. A triangle, S, with area 5 square units is transformed by $\mathbf{M}$ into triangle T.

    Triangle U is obtained by rotating triangle S 135° anticlockwise about the origin.

    (a) Find the area of triangle T.

    (b) Find the matrix that transforms triangle T into triangle S.

    (c) Find the matrix which transforms triangle S into triangle U.

    (d) Find the matrix which transforms triangle T into triangle U.

29. The plane is transformed by the matrix $\mathbf{M} = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$.

    (a) Show that $\det \mathbf{M} = 0$, and that the whole plane is mapped onto the line $x - 2y = 0$.

    (b) The points $P(x, y)$ is mapped to $P'(4, 2)$. Show that P could be anywhere on the line $x + 2y = 2$

    (c) Find the equation of the line of points that map to $(10, 5)$

30. (a) Write down the matrices representing:

    i. A rotation of 90° anticlockwise about the origin ($\boldsymbol{A}$).

    ii. A reflection in the line $y = x$ ($\boldsymbol{B}$).

    (b) Does $\boldsymbol{AB} = \boldsymbol{BA}$?

    (c) Describe the transformations $\boldsymbol{AB}$ and $\boldsymbol{BA}$ geometrically.

    (d) Write down expressions for $\boldsymbol{A}^{-1}$ and $\boldsymbol{B}^{-1}$

3.2 Level 2

31. (a) Describe the geometric effect of the transformation represented by the matrix $\boldsymbol{P}$

$$\boldsymbol{P} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

    (b) Hence write down an expression for $\boldsymbol{P}^4$

    (c) Find the value of $\det(\boldsymbol{P})$ and explain this result geometrically.

32. (a) Show that $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$ algebraically.

    (b) Give a geometric explanation of this result.

33. The trace, T of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as $a + d$

(a) Prove that $T(\boldsymbol{AB}) = T(\boldsymbol{BA})$

(b) Prove that $T(\boldsymbol{A}+\boldsymbol{B}) = T(\boldsymbol{B}+\boldsymbol{A})$

34. This question is about proof by induction with matrices: strictly this is extension material at the moment but you're welcome to try it:

(a) If $\boldsymbol{M} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ prove that

$$\boldsymbol{M^2} = 3\boldsymbol{M} - 2\boldsymbol{I}$$

(b) Prove by induction that

$$\boldsymbol{M^n} = (2^n - 1)\boldsymbol{M} - 2\left(2^{n-1} - 1\right)\boldsymbol{I}$$

4 Trigonometry

4.1 Level 1

35. Write down the exact values of the following without using a machine of evil (calculator):

(a) $\sin 75 \cos 75$

(b) $2\cos^2 22.5 - 1$

(c) $\cos^2 35 + \sin^2 35$

(d) $\frac{1-\tan^2 15}{2\tan 15}$

36. Solve for $0 \leq \theta < 2\pi$ (without a caluculator)

(a) $\tan\theta = \frac{1}{\sqrt{3}}$

(b) $\cos 4\theta = -\frac{1}{2}$

(c) $\sin\theta - \sqrt{3}\cos\theta = 0$

(d) $2\cos^2\theta + 3\sin\theta - 3 = 0$

37. Prove the following identities:

(a)

$$\cot x - \cot 2x \equiv \text{cosec}2x$$

(b)

$$\frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta - \sin\theta} \equiv \frac{2\cos\theta}{\cos 2\theta}$$

(c)

$$\tan^2\frac{\theta}{2} \equiv \frac{1-\cos\theta}{1+\cos\theta}$$

(d) *

$$4\cos^2\frac{A}{2} \equiv \frac{1-\cos 2A}{1-\cos A}$$

(e) **

$$\frac{\sec^2\theta + 2\tan\theta}{(\cos\theta + \sin\theta)^2} \equiv \sec^2\theta$$

38. Given that θ is an acute angle and $\sin\theta = \frac{4}{5}$, find the value of each of:
    (a) $\sin 2\theta$
    (b) $\cos 2\theta$
    (c) $\tan 2\theta$

39. Solve each of the following equations for $0 \leq \theta \leq 360$
    (a) $3\sin 2\theta = \sin\theta$
    (b) $3\cos 2\theta - \cos\theta + 2 = 0$
    (c) $6\cos 2\theta + 6 = 7\sin\theta$
    (d) $3\cos\frac{\theta}{2} = 2\sin\theta$

40. A sector of a circle of radius 5cm subtends an angle of $\frac{3\pi}{10}$ at the centre. Calculate
    (a) the length of the arc of the sector
    (b) the area of the sector.

41. Calculate the area of a segment of angle $\frac{\pi}{3}$ cut from a circle of radius 10cm.

4.2 Level 2

42. **Caution: spicy.** The following diagram shows a taut belt passing around two circles which touch at P. The smaller circle has radius 1, the larger circle has center B and radius 3.
    (a) Find the total length of the belt.
    (b) Find an exact expression for the shaded area.
    (c) Generalize to circles of radius r_1 and r_2.
    (d) The following diagram shows a taut belt passing around two circles. The distance between the centres of the two circles, AB is equal to d Find an expression for the length of the belt and the shaded area in terms of r_1, r_2 and the distance d

5 Curve sketching and inequalities

5.1 Level 1

43. Sketch the following curves making sure to show intercepts, turning points and long term behaviour where appropriate. For questions f to j make sure you mark on any points of intersection between the two curves.

(a) $y = x(3 + x)$

(b) $y = (x - 2)(3 + x)$ (no need for turning point)

(c) $y = (2x + 1)(x - 2)(x + 6)$ (no need for turning points)

(d) $y = x^2(3 + x)$ (no need for turning points)

(e) $y = (x - 2)^2 - 5$ (no need for intercepts)

(f) $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ on the same axes

(g) $y = 2^x$ and $y = 3^x$ on the same axes

(h) $y = \frac{1}{x}$ and $y = \frac{1}{x+3}$ and $y = \frac{1}{x} + 2$ on the same axes

(i) $y = \sin(x)$ and $y = \sin(3x)$ and $y = 2\sin(x)$ on the same axes

(j) $y = \cos(x)$ and $y = \cos(\frac{x}{2})$ and $y = -\cos(x)$ on the same axes

44. (a) Sketch the curve

$$y = \frac{x-4}{x^2-4}$$

(b) With reference to your sketch, solve the inequality

$$\frac{x-4}{x^2-4} < 1$$

5.2 Level 2

45. (a) Sketch the curve

$$y = \frac{x^2-3x+3}{x^2-4x+3}$$

(b) With reference to your sketch, solve the inequality

$$\left|\frac{x^2-3x+3}{x^2-4x+3}\right| \leq 1$$

NB: the modulus function will not be examined in the upcoming assessments, except in the context of complex numbers.

46. Find the range of values of the curve

$$y = \frac{x^2+2x+3}{x^2+3x+2}$$

Hence sketch the curve.

47. Find the range of values of the curve

$$y = \frac{(x-1)^2}{x^2+x+1}$$

Hence sketch the curve.

48. Sketch the locus of points satisfying

$$y^2 = \frac{2x}{x^2 - 1}$$

49. Find the range of values of the function

$$y = \frac{(x - \lambda)^2}{x^2 + 2x + 5}$$

Giving your answer in terms of λ.

50. The function

$$\frac{x - \lambda}{x^2 - 2x + \lambda}$$

takes all real values. Find the range of values of λ

6 Summation of Series

6.1 Level 1

Make sure you have memorised the formulae for the sum of an arithmetic and geometric sequence.

51. Evaluate $1 + 7 + 13 + ... + 337$

52. Evaluate $3 + 6 + 12 + ... + 1536$

53. Evaluate $1 - \frac{2}{3} + \frac{4}{9} - ...$

54. Evaluate

(a)
$$\sum_{i=5}^{15} u_i$$
where $u_{i+1} = u_i - 3$ and $u_1 = 57$

(b)
$$\sum_{r=1}^{6} \sqrt{18 \cdot 2^r}$$

(c)
$$\sum_{r=1}^{4} \frac{a}{3^r}$$
given that:
$$\sum_{r=1}^{\infty} \frac{a}{3^r} = 4$$

55. For what values of n is is true that:
$$\sum_{i=1}^{n} 33 - 2i > 200$$

56. Find a simplified expression for the sum
$$\frac{a}{k^4} + \frac{a}{k^3} + \frac{a}{k^2} + \frac{a}{k} + a + ak + ak^2 + ak^3 + ak^4$$

57. Use known results to sum
$$\sum_{k=1}^{n} (2k-1)^2$$

58. (a) By using the formulas for the standard series find and simplify the following sum
$$\sum_{k=1}^{n} (k(k+1)(k+2))$$

(b) * Obtain the same result by using a telescoping series

59. (a) Simplify

$$\frac{1}{2r+1} - \frac{1}{2r+3}$$

(b) Hence find the sum

$$\frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \cdots \frac{1}{99 \times 101}$$

60. By looking for a suitable telescoping series find the following series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

6.2 Level 2

61. By looking for a suitable telescoping series find the following series

$$\sum_{k=1}^{\infty} \frac{2k+1}{k^2\left(k+1\right)^2}$$

62. Given that

$$\arctan(x+1) - \arctan(x-1) = \arctan \frac{2}{x^2}$$

deduce the sum of the series

$$\arctan \frac{2}{1^2} + \arctan \frac{2}{2^2} + \arctan \frac{2}{3^2} + \cdots + \arctan \frac{2}{n^2}$$

63. By looking for a suitable telescoping series find the following series

$$\sum_{k=r}^{\infty} \frac{1}{\binom{k}{r}}$$

64. **Caution: spicy extension** It's fun but make sure you have done everything else first!

(a) Find a simplified expression for

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

(b) Find a simplified expression for

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} \cdots + n\binom{n}{n}$$

(c) Find a simplified expression for

$$\binom{2n}{0}^2 + \binom{2n}{1}^2 + \binom{2n}{3}^2 + \cdots + \binom{2n}{n}^2$$

(d) Find a simplified expression for

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k}$$

7 Induction

Prove the following results by induction.

65. Prove that

$$1.1! + 2.2! + 3.3! + \ldots n.n! = (n+1)! - 1$$

66. Prove that

$$1^2 + 3^2 + 5^2 + \ldots (2n-1)^2 = \frac{1}{3}n\left(4n^2 - 1\right)$$

67. Prove that

$$1^4 + 2^4 + 3^4 + \ldots n^4 = \frac{1}{30}n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)$$

68. A sequence is defined by $a_1 = 7$ and $a_{n+1} = 7a_n - 3$. Prove that:

$$a_n = \frac{13 \times 7^{n-1} + 1}{2}$$

69. A sequence is defined by $u_1 = 2$ and $u_{n+1} = \frac{u_n}{1+u_n}$. Prove that:

$$u_n = \frac{2}{2n-1}$$

70. A sequence is defined by $u_1 = 2$ and $u_{n+1} = \frac{u_n+1}{2}$. Conjecture an ordinal formula and prove your result by induction.

71. Prove that, for $n \geq 2$

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\ldots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$$

8 Answers

1. (a) $5 - i$
   (b) $2x^2 - 2y^2 + 5xyi$
   (c) $p^2 + 4q^2$
   (d)
$$i^n = \begin{cases} 1 & \text{when n is a multiple of 4} \\ i & \text{when n is one more than a multiple of 4} \\ -1 & \text{when n is two more than a multiple of 4} \\ -i & \text{when n is three more than a multiple of 4} \end{cases}$$
   (e) $\frac{2-3i}{13}$
   (f) $\frac{4+7i}{5}$

2. (a) $\sqrt{5}$ and $\arctan \frac{1}{2}$
   (b) $\sqrt{5}$ and $\pi - \arctan 2$
   (c) 3 and $\frac{\pi}{2}$
   (d) 1 and $-\theta$
   (e) 1 and $\frac{\pi}{2} - \theta$

3. (a) $z = -\frac{8+6i}{5}$
   (b) $\frac{3-11i}{26}$
   (c) $z = 2 \pm 5i$
   (d) $z = 2i, -3i$

   Roots in conjugate pairs $\iff$ coefficients real.

4. (a) $|z| = 5$, $\arg(z) = \arctan(\frac{4}{3}) = 53.1^o$
   (b) $|w| = 13$, $\arg(w) = \arctan(\frac{5}{12}) = 22.6^o$
   (c) $|\frac{1}{z}| = \frac{1}{5}$, $\arg(\frac{1}{z}) = -53.1^o$
   (d) $|\frac{1}{w}| = \frac{1}{13}$, $\arg(\frac{1}{w}) = -22.6^o$
   (e) $|zw| = |z||w| = 65$, $\arg(zw) = \arg(z) + \arg(w) = 75.7^o$
   (f) $|z^*| = 5$, $\arg(z^*) = -\arg(z) = -53.1^o$
   (g) $|w^*| = 13$, $\arg(w^*) = -\arg(w) = -22.6^o$
   (h) $|(zw)^*| = |z||w| = 65$, $\arg(zw)^* = -75.7^o$
   (i) $|w^2| = 169$, $\arg(w^2) = 2\arg(w) = 45.2^o$
   (j) $|z^{10}| = 5^{10}$, $\arg(z^{10}) = 10\arg(z) = 10 \times \arctan(\frac{4}{3}) = 531.30..^o$ which is the same argument as 171.3^o (1 d.p.)

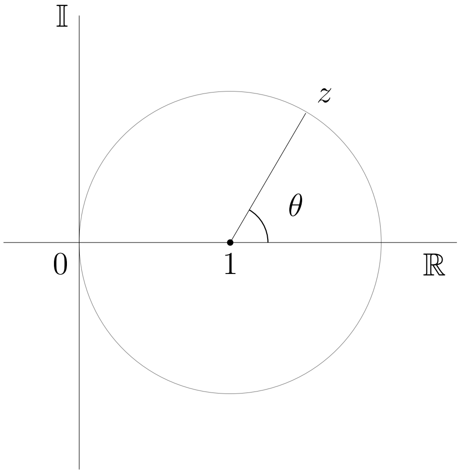
5. both equal to $a^2 + b^2$

   both equal to $\frac{1}{\sqrt{a^2+b^2}}$

6. $(a+bi)(a-bi) \equiv a^2+b^2 \equiv |z|^2$
7. (a) $z = 1 \pm i, -1, 3$
   (b) $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$
8. (a)

$$\begin{aligned} && 1 &= |z|^2 \\ \Rightarrow && 1 &= zz^* \\ \Rightarrow && \frac{1}{z^*} &= z \end{aligned}$$

   (b) If $|z| = 1$ then $z+1$ is a circle with radius 1 and centre 1, as illustrated below.



   This lies in the second and fourth quadrant.

9. $-16\sqrt{3}\, i$
10. (a) $\sqrt{2}$ and $\frac{\pi}{4}$
    (b) $-2^{50}(1+i)$
11. $x^3 - 9x^2 + 26x - 24$
12. $(9 + 6x + 7x^2)(x-1) + 5$
13.

$$x^4 + 1 \equiv \left(x^2 - \sqrt{2}x + 1\right)\left(x^2 + \sqrt{2}x + 1\right)$$

    Roots are $\pm\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$
14. $k = -1$ or $k = -\frac{4}{3}$
15. Calling the roots α, α and β, Vieta gives:

$$\begin{aligned} \alpha + \alpha + \beta &= 0 \\ 2\alpha\beta + \alpha^2 &= p \\ \alpha^2\beta &= -q \end{aligned}$$

Using $\beta = 2\alpha$ in the other equations gives $p = -3\alpha$ and $q = -2\alpha^3$. $4p^3 = -108\alpha^6$ and $27q^2 = 108\alpha^6$ and the result follows.

16. Roots are $-4, -2, 2$ and 3

17. (a) $2x^2 - 7 = 0$
    (b) $2x^2 + 12x - 45 = 0$
    (c) $5x^2 + 18x + 5 = 0$

18. (a) $2x^3 - 3x^2 - 13x + 7 = 0$
    (b) $2x^3 + 15x^2 + 23x - 5 = 0$
    (c) $x^3 + 3x^2 - 26x - 28 = 0$
    (d) $7x^3 + 13x^2 - 3x - 2 = 0$

19. (a) roots are $-4, 3, -3$
    (b) roots are $-2, 5, -4$
    (c) roots are $-2, \frac{3}{4}, \frac{3}{4}$
    (d) roots are $\frac{3}{2}, \frac{3+\sqrt{5}}{4}, \frac{3-\sqrt{5}}{4}$
    (e) roots are $3, 6, 12$
    (f) roots are $3, 5, 7$

20. (a) i. 6
        ii. 5
    (b) $x^2 - 6x + 5 = 0$
    (c) Using $x^2 + y^2 = (x+y)^2 - 2xy = 26$ gives diagonal is $\sqrt{26}$ cm.

21. (a) $x^3 - 25x^2 + 40x - 10$
    (b) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta\alpha\gamma + \beta\gamma)$ to get $\alpha^2 + \beta^2 + \gamma^2 = 545$. So the space diagonal is $\sqrt{545}$

22. (a) 2
    (b)

$$\begin{aligned}\left(\alpha^3 + \beta^3 + \gamma^3\right) &= (\alpha + \beta + \gamma)\left(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma\right) + 3\alpha\beta\gamma \\ &= 2(2-1) + 3 \times 6 \\ &= 20\end{aligned}$$

23. Expand the brackets. N.B¿ these are what you used in question 20.

24. Let

$$P(x) = \frac{x^2 - a^2}{(a-b)(a-c)} + \frac{x^2 - b^2}{(b-c)(b-a)} + \frac{x^2 - c^2}{(c-a)(c-b)} + 1$$

$$
\begin{aligned}
P(a) &= \frac{a^2-b^2}{(b-c)(b-a)} + \frac{a^2-c^2}{(c-a)(c-b)} + 1 \\
&= \frac{a+b}{c-b} - \frac{a+c}{c-b} + 1 \\
&= 0
\end{aligned}
$$

By a similar argument $P(a) = P(b) = P(c) = 0$, however $P(x)$ is a quadratic expression, and a quadratic expression with three roots must be identically zero.

25. $3a^2 + 3b^2 + 3c^2 - 2ab - 2ac - 2bc$

26. (a) $3A = \begin{pmatrix} 0 & -3 \\ 3 & 0 \end{pmatrix}$

    (b) $AB = \begin{pmatrix} -2 & -3 \\ 2 & 1 \end{pmatrix}$

    (c) $AC = \begin{pmatrix} -3 & -2 & -1 \\ 1 & 2 & 3 \end{pmatrix}$

    (d) CA doesn't exist

    (e) $A + C$ doesn't exist

    (f) $A + B = \begin{pmatrix} 2 & 0 \\ 3 & 3 \end{pmatrix}$

27. (a) $\begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$

    (b) $-\frac{1}{2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$

    (c) $\frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

    (d) $\begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix}$

    (e) $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$

28. (a) 150 square units.

    (b) $\frac{1}{30}\begin{pmatrix} 6 & 3 \\ 4 & 7 \end{pmatrix}$

    (c) $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

    (d) $\frac{1}{15\sqrt{2}}\begin{pmatrix} -5 & -5 \\ 1 & 2 \end{pmatrix}$

29. part (c) $x + 2y = 5$

30. (a)

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
$$\boldsymbol{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(b)

$$\boldsymbol{AB} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\boldsymbol{BA} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(c) $\boldsymbol{AB}$ is a reflection in the y-axis. $\boldsymbol{BA}$ is a reflection in the x-axis.

(d)

$$\boldsymbol{A}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\boldsymbol{B}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

31. (a) Rotation by $\frac{\pi}{2}$ with centre $(0,0)$ and enlargement by a scale factor of 2
(b) $16\boldsymbol{I}$
(c) $\det(\boldsymbol{A}) = 4$. This is the area scale factor.

32. Area scale factors are multiplicative.

33. If $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$
(a) Both equal $ap + br + cq + ds$
(b) Both equal $a + p + d + s$

34. Use part a) for the induction step.

35. (a) $\frac{1}{4}$
(b) $\frac{1}{\sqrt{2}}$
(c) 1
(d) $\sqrt{3}$

36. (a) $\theta = \frac{\pi}{6}, \frac{7\pi}{6}$
(b) $\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{11\pi}{6}$

(c) $\theta = \frac{\pi}{3}, \frac{4\pi}{3}$

(d) $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

37. (a)

$$\begin{aligned}
\cot x - \cot 2x &\equiv \frac{\cos x}{\sin x} - \frac{\cos 2x}{\sin 2x} \\
&\equiv \frac{2\cos^2 x}{2\sin x \cos x} - \frac{\cos 2x}{\sin 2x} \\
&\equiv \frac{\cos(2x) + 1}{\sin 2x} - \frac{\cos 2x}{\sin 2x} \\
&\equiv \frac{1}{\sin 2x}
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta - \sin\theta} &\equiv \frac{\cos\theta - \sin\theta}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)} + \frac{\cos\theta - \sin\theta}{(\cos\theta + \sin\theta)(\cos\theta + \sin\theta)} \\
&\equiv \frac{2\cos\theta}{\cos^2\theta - \sin^2\theta} \\
&=\equiv \frac{2\cos\theta}{\cos 2\theta}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{1 - \cos\theta}{1 + \cos\theta} &\equiv \frac{1 - \left(1 - 2\sin^2\frac{\theta}{2}\right)}{1 + \left(2\cos^2\frac{\theta}{2} - 1\right)} \\
&\equiv \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} \\
&\equiv \tan^2\frac{\theta}{2}
\end{aligned}$$

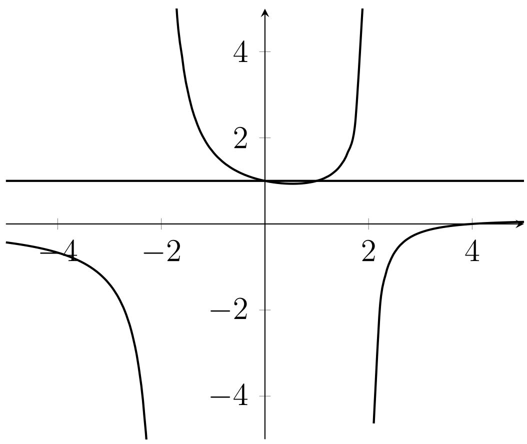
(d)

$$\begin{aligned}
\frac{1 - \cos 2A}{1 - \cos A} &\equiv \frac{1 - \left(1 - 2\sin^2 A\right)}{1 - \left(1 - 2\sin^2\frac{A}{2}\right)} \\
&\equiv \frac{\sin^2 A}{\sin^2\frac{A}{2}} \\
&\equiv \frac{\left(2\cos\frac{A}{2}\sin\frac{A}{2}\right)^2}{\sin^2\frac{A}{2}} \\
&=\equiv 4\cos^2\frac{A}{2}
\end{aligned}$$

(e)

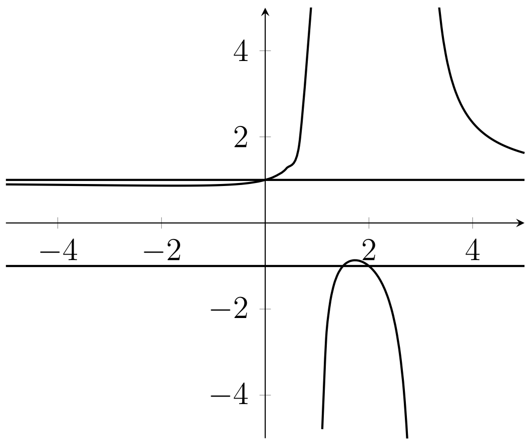
$$\begin{aligned}
\frac{\sec^2\theta + 2\tan\theta}{(\cos\theta+\sin\theta)^2} &\equiv \frac{\sec^2\theta + 2\tan\theta}{\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta} \\
&\equiv \frac{\sec^2\theta\left(1+2\frac{\sin\theta}{\cos\theta}\cos^2\theta\right)}{1+2\sin\theta\cos\theta} \\
&\equiv \frac{\sec^2\theta\,(1+2\sin\theta\cos\theta)}{1+2\sin\theta\cos\theta} \\
&\equiv \sec^2\theta
\end{aligned}$$

38. (a) $\frac{24}{25}$
    (b) $-\frac{7}{25}$
    (c) $-\frac{24}{7}$
39. (a) $0°, 80.4°, 180°, 279.6°, 360°$
    (b) $60°, 109.5°, 250.5°, 300°$
    (c) $131.4°, 48.6°$
    (d) $97.2°, 180°, 262.8°$
40. (a) $\frac{3\pi}{2}$cm
    (b) $\frac{15\pi}{4}$cm^2
41. $\frac{50\pi}{3}$cm^2
42. (a) $4\sqrt{3}+\frac{14}{3}\pi$
    (b)
    (c)
    (d)
43. Use Desmos to check.
44. (a)



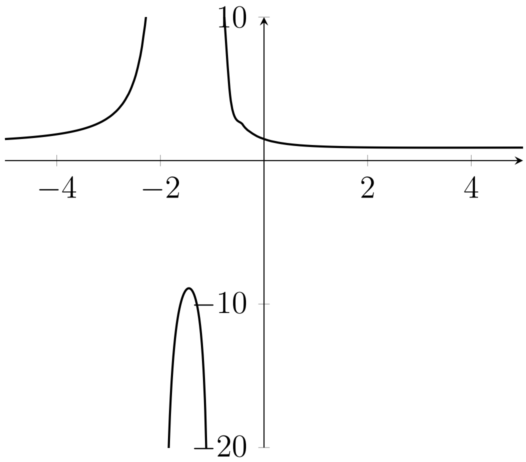
    (b) $x < -2$, $0 < x < 1$ and $x > 2$

45. (a)



(b) $x \leq 0$ or $\frac{3}{2} \leq x \leq 2$

46. $y \leq -4 - 2\sqrt{6}$ or $y \geq -4 - 2\sqrt{6}$



For a non-calculus method of finding the range, write the equation of the curve as:

$$x^2(y-1) + (3y-2)x + (2y-3) = 0$$

This has real-solutions for x when the discriminant is non-negative, so $y^2 + 8y - 8 \geq 0$

47. For the range of values, write the equation as

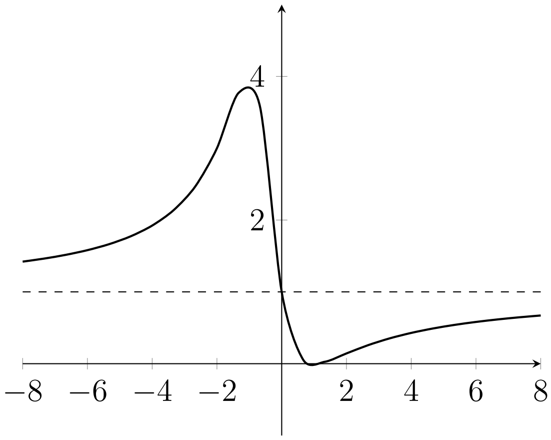
$$x^2(y-1) + (y+2)x + (y-1) = 0$$

This has real solutions for x when the discriminant is non-negative, so $12y - 3y^2 \geq 0$. This means

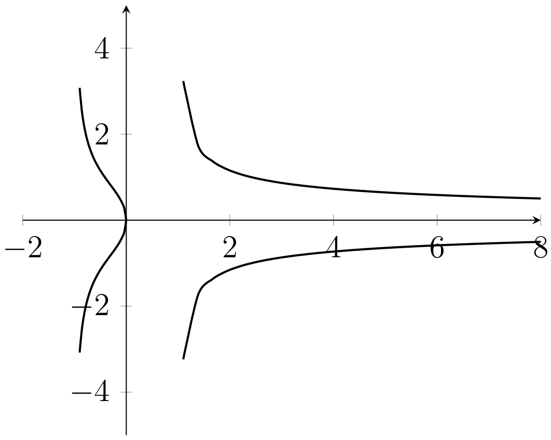
$$0 \leq y \leq 4$$

.

It is easy to show that $y = 0$ when $x = 1$ and $y = 4$ when $x = -1$. There are no vertical asymptotes, and there is a horizontal asymptote of $y = 1$



48. Firstly note that $\dfrac{2x}{x^2-1} \geq 0$ only if $-1 < x < 1$ or if $x > 1$. As we can take positive or negative values of y the curve is symmetric in the x-axis.



49. Firstly write in the form

$$x^2(y-1) + (2y + 2\lambda)\, x + 5y - \lambda^2 = 0$$

This has real solutions for x if the discriminant is non-negative so

$$\left(20 + 8\lambda + 4\lambda^2\right) y - 16y^2 \geq 0$$

We thus have

$$0 \leq y \leq \frac{5 + 2\lambda + \lambda^2}{4}$$

50. As before write in the form

$$yx^2 - (2y+1)\, x + \lambda\, (y+1) = 0$$

This has real solutions if the discriminant is non-negative so:

$$4\,(1-\lambda)\, y^2 + 4(1-\lambda)y + 1 \geq 0$$

This quadratic expression will always be positive if the x^2 coefficient is positive and the discriminant is non-negative. So we want $\lambda < 1$ and $16\lambda^2 - 16\lambda \leq 0$, so $0 < \lambda < 1$. Both these conditions are satisfied in the region $0 \leq \lambda < 1$, but we must discount $\lambda = 0$ as a factor cancels.

51. 9633

52. 3069

53. $\frac{3}{5}$

54. (a) 330

    (b) $42 + 42\sqrt{2}$

    (c) $\frac{320}{81}$

55.

56.

$$\frac{a\left(1+k+k^2\right)\left(1+k^3+k^6\right)}{k^4}$$

57.

$$\frac{n\left(4n^2-1\right)}{3}$$

58. (a)

$$\frac{1}{4}n(n+1)(n+2)(n+3)$$

    (b) The teloscoping is

$$\frac{n(n+1)(n+2)(n+3)}{4} - \frac{(n-1)n(n+1)(n+2)}{4} = n(n+1)(n+2)$$

59. (a) $\dfrac{2}{(2r+1)\,(2r+3)}$

    (b) $\dfrac{50}{101}$

60. $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$ and so sum is 1.

61. $\dfrac{2k+1}{k^2(k+1)^2} = \dfrac{1}{k^2} - \dfrac{1}{(k+1)^2}$ and so sum is 1.

62. The sum is

$$\arctan\left(n+1\right) + \arctan n - \frac{\pi}{4}$$

63. The telescoping is

$$\begin{aligned}\frac{1}{\binom{k}{r}} &= \frac{r!}{(k+1-r)(k+2-r)\dots(k-1)k} \\ &= \frac{r!}{r-1}\left(\frac{1}{(k+1-r)\dots(k-1)} - \frac{1}{(k+2-r)(k+2-r)\dots(k-1)\,k}\right)\end{aligned}$$

So the sum to infinity is

$$\frac{r!}{(r-1)!\,(r-1)} = \frac{r}{r-1}$$

64. (a) The answer is 2^n one easy way to show this is to use the binomial expansion:

$$(1+1)^n = 1 + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}$$

(b) If we start with

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n$$

Then differentiating both sides with respect to x give

$$n\,(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2} x + \cdots + n\binom{n}{n} x^{n-1}$$

Putting $x = 1$ gives the desired sum as $n2^{n-1}$

(c)

$$\frac{\binom{4n}{2n} + \binom{2n}{n}^2}{2}$$

The best way to see this to consider picking $2n$ people from a group of $4n$ people consisting of $2n$ people with hats and $2n$ people without. Counting arrangements gives

$$\binom{4n}{2n} = \binom{2n}{0}\binom{2n}{2n} + \binom{2n}{1}\binom{2n}{2n-1} + \binom{2n}{2}\binom{2n}{2n-2} + \ldots \binom{2n}{2n}\binom{2n}{0}$$

Where the first term represents ways of choosing $2n$ with hats and 0 without hats and so forth. Using symmetry gives:

$$\binom{4n}{2n} = \binom{2n}{0}^2 + \binom{2n}{1}^2 + \binom{2n}{2}^2 + \ldots \binom{2n}{2n}^2$$

Again using symmetry we get:

$$\binom{4n}{2n} = 2\binom{2n}{0}^2 + 2\binom{2n}{1}^2 + 2\binom{2n}{2}^2 + \cdots + 2\binom{2n}{n-1}^2 + \binom{2n}{n}^2$$

Noting that we pick up two copies of all but the middle term. Rearranging gives the result.

(d) $\binom{n+1}{k+1}$- best way is to consider picking $k+1$ objects from $n+1$ objects, and condition on the position of the first element. e.g. if you pick element 1 then you have to pick k from the remaining n, but if the first object you pick is object two then you need to choose k objects from the remaining $n-1$ and so on.