Polar Curves and Calculus with Inverse Trig – Past Paper Questions

1.

(a) A curve has polar equation $r = ae^{-k\theta}$ for $0 \leqslant \theta \leqslant \pi$, where a and k are positive constants. The points A and B on the curve correspond to $\theta = 0$ and $\theta = \pi$ respectively.

(i) Sketch the curve. [2]

(ii) Find the area of the region enclosed by the curve and the line AB. [4]

(b) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{3+4x^2}\,dx$. [5]

2.

(a) A curve has polar equation $r = a(1 - \cos\theta)$, where a is a positive constant.

(i) Sketch the curve. [2]

(ii) Find the area of the region enclosed by the section of the curve for which $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ and the line $\theta = \frac{1}{2}\pi$. [6]

(b) Use a trigonometric substitution to show that $\displaystyle\int_0^1 \frac{1}{\left(4 - x^2\right)^{\frac{3}{2}}}\,dx = \frac{1}{4\sqrt{3}}$. [4]

(c) In this part of the question, $f(x) = \arccos(2x)$.

(i) Find $f'(x)$. [2]

3.

(a) Fig. 1 shows the curve with polar equation $r = a(1 - \cos 2\theta)$ for $0 \leqslant \theta \leqslant \pi$, where a is a positive constant.

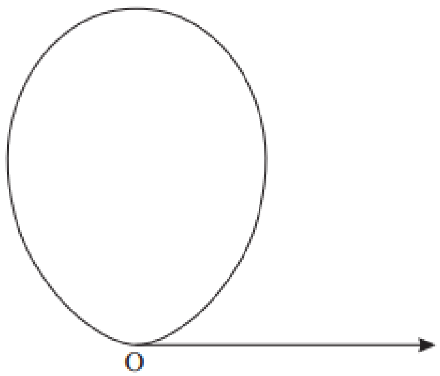


Fig. 1

Find the area of the region enclosed by the curve. [7]

(b) **(i)** Given that $f(x) = \arctan(\sqrt{3} + x)$, find $f'(x)$ and $f''(x)$. [4]

4.

(a) A curve has cartesian equation $(x^2 + y^2)^2 = 3xy^2$.

(i) Show that the polar equation of the curve is $r = 3\cos\theta\sin^2\theta$. [3]

(ii) Hence sketch the curve. [3]

(b) Find the exact value of $\displaystyle\int_0^1 \frac{1}{\sqrt{4 - 3x^2}}\,\mathrm{d}x$. [5]

5.

(i) Given that $y = \arctan\left(\frac{x}{a}\right)$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{a^2 + x^2}$. **[4]**

(ii) Find the exact values of the following integrals.

(*A*) $\int_{-2}^{2} \frac{1}{4 + x^2}\,\mathrm{d}x$ **[3]**

(*B*) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{1 + 4x^2}\,\mathrm{d}x$ **[3]**

6.

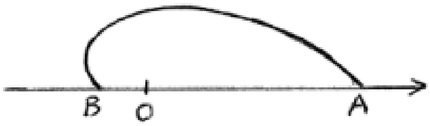
(a) A curve has polar equation $r = a \tan \theta$ for $0 \leqslant \theta \leqslant \frac{1}{3}\pi$, where a is a positive constant.

(i) Sketch the curve. **[3]**

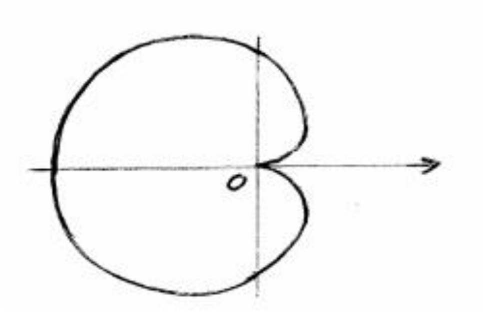
(ii) Find the area of the region between the curve and the line $\theta = \frac{1}{4}\pi$. Indicate this region on your sketch. **[5]**

Polar Curves and Calculus with Inverse Trig – Past Paper Answers

7.

1(a)(i)		B1 B1	**2**	Correct shape for $0 \le \theta \le \frac{1}{2}\pi$ Correct shape for $\frac{1}{2}\pi \le \theta \le \pi$ *Requires decreasing r on at least one axis* *Ignore other values of* θ
(ii)	Area is $\int \frac{1}{2} r^2 \, d\theta = \int_0^{\pi} \frac{1}{2} a^2 (e^{-k\theta})^2 \, d\theta$ $= \left[-\frac{a^2}{4k} e^{-2k\theta} \right]_0^{\pi}$ $= \frac{a^2}{4k}(1 - e^{-2k\pi})$	M1 A1 M1 A1	**4**	For $\int (e^{-k\theta})^2 \, d\theta$ For a correct integral expression including limits *(may be implied by later work) (Condone reversed limits)* Obtaining a multiple of $e^{-2k\theta}$ as the integral
(b)	$\int_0^{\frac{1}{2}} \frac{1}{3 + 4x^2} \, dx = \left[\frac{1}{2\sqrt{3}} \arctan\left(\frac{2x}{\sqrt{3}} \right) \right]_0^{\frac{1}{2}}$ $= \frac{1}{2\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}} \right)$ $= \frac{\pi}{12\sqrt{3}}$	M1 A1A1 M1 A1	**5**	For arctan For $\frac{1}{2\sqrt{3}}$ and $\frac{2x}{\sqrt{3}}$ *Dependent on first M1*
	OR Putting $2x = \sqrt{3} \tan \theta$ Integral is $\int_0^{\frac{1}{6}\pi} \frac{1}{2\sqrt{3}} \, d\theta$ $= \frac{\pi}{12\sqrt{3}}$	M1 A1 A1 M1 A1		For any tan substitution For $\int \frac{1}{2\sqrt{3}} \, d\theta$ For changing to limits of θ *Dependent on first M1*

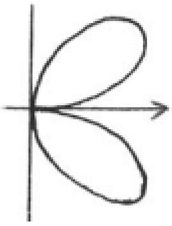
8.

(a)(i)		B2	2	Must include a sharp point at O and have infinite gradient at $\theta = \pi$ Give B1 for r increasing from zero for $0 < \theta < \pi$, or decreasing to zero for $-\pi < \theta < 0$
(ii)	Area is $\int \frac{1}{2}r^2\,d\theta = \int_0^{\frac{1}{2}\pi} \frac{1}{2}a^2(1-\cos\theta)^2\,d\theta$	M1 A1		For integral of $(1-\cos\theta)^2$ For a correct integral expression including limits *(may be implied by later work)*
	$= \frac{1}{2}a^2 \int_0^{\frac{1}{2}\pi} \left(1 - 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\right) d\theta$	B1		Using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
	$= \frac{1}{2}a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta \right]_0^{\frac{1}{2}\pi}$	B1B1 ft		Integrating $a + b\cos\theta$ and $k\cos 2\theta$
	$= \frac{1}{2}a^2(\frac{3}{4}\pi - 2)$	B1	6	Accept $0.178a^2$
(b)	Put $x = 2\sin\theta$	M1		or $x = 2\cos\theta$
	Integral is $\int_0^{\frac{1}{6}\pi} \frac{1}{(4-4\sin^2\theta)^{\frac{3}{2}}}(2\cos\theta)\,d\theta$	A1		Limits not required
	$= \int_0^{\frac{1}{6}\pi} \frac{2\cos\theta}{8\cos^3\theta}\,d\theta = \int_0^{\frac{1}{6}\pi} \frac{1}{4}\sec^2\theta\,d\theta$			
	$= \left[\frac{1}{4}\tan\theta \right]_0^{\frac{1}{6}\pi}$	M1		For $\int \sec^2\theta\,d\theta = \tan\theta$
	$= \frac{1}{4} \times \frac{1}{\sqrt{3}} = \frac{1}{4\sqrt{3}}$	A1 ag	4	*SR* If $x = 2\tanh u$ is used M1 for $\frac{1}{4}\sinh(\frac{1}{2}\ln 3)$ A1 for $\frac{1}{8}(\sqrt{3} - \frac{1}{\sqrt{3}}) = \frac{1}{4\sqrt{3}}$ (max 2 / 4)
(c)(i)	$f'(x) = \frac{-2}{\sqrt{1-4x^2}}$	B2	2	Give B1 for any non-zero real multiple of this (or for $\frac{-2}{\sin y}$ etc)

9.

1(a)	Area is $\int_0^{\pi} \frac{1}{2}a^2(1-\cos 2\theta)^2\,\mathrm{d}\theta$ $= \int_0^{\pi} \frac{1}{2}a^2\left(1-2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right)\mathrm{d}\theta$ $= \frac{1}{2}a^2\left[\frac{3}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_0^{\pi}$ $= \frac{3}{4}\pi a^2$	M1 A1 B1 B1B1B1 ft A1 **7**	For $\int(1-\cos 2\theta)^2\,\mathrm{d}\theta$ Correct integral expression including limits (may be implied by later work) For $\cos^2 2\theta = \frac{1}{2}(1+\cos 4\theta)$ Integrating $a+b\cos 2\theta+c\cos 4\theta$ *[Max B2 if answer incorrect and no mark has previously been lost]*
(b)(i)	$\mathrm{f}'(x) = \frac{1}{1+(\sqrt{3}+x)^2}$ $\mathrm{f}''(x) = \frac{-2(\sqrt{3}+x)}{\left(1+(\sqrt{3}+x)^2\right)^2}$	M1 A1 M1 A1 **4**	Applying $\frac{\mathrm{d}}{\mathrm{d}u}\arctan u = \frac{1}{1+u^2}$ or $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y}$ Applying chain (or quotient) rule

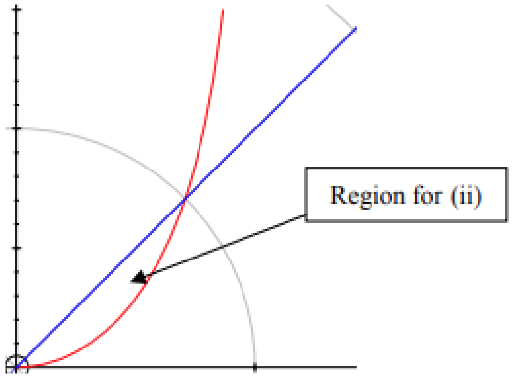
10.

I(a)(i)	$x = r\cos\theta,\ y = r\sin\theta$ $(r^2\cos^2\theta + r^2\sin^2\theta)^2 = 3(r\cos\theta)(r\sin\theta)^2$ $r^4 = 3r^3\cos\theta\sin^2\theta$ $r = 3\cos\theta\sin^2\theta$	M1 A1 A1 ag **3**	(M0 for $x = \cos\theta,\ y = \sin\theta$)
(ii)		B1 B1 B1 **3**	Loop in 1st quadrant Loop in 4th quadrant Fully correct curve *Curve may be drawn using continuous or broken lines in any combination*
(b)	$\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \left[\frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}\,x}{2}\right]_0^1$ $= \frac{1}{\sqrt{3}}\arcsin\frac{\sqrt{3}}{2}$ $= \frac{\pi}{3\sqrt{3}}$	M1 A1A1 M1 A1 **5**	For arcsin For $\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{3}\,x}{2}$ Exact numerical value *Dependent on first M1* (M1A0 for $60/\sqrt{3}$)
	OR Put $\sqrt{3}\,x = 2\sin\theta$ — M1 A1 $\int_0^1 \frac{1}{\sqrt{4-3x^2}}\,dx = \int_0^{\frac{\pi}{3}} \frac{1}{\sqrt{3}}\,d\theta$ — A1 $= \frac{\pi}{3\sqrt{3}}$ — M1A1		Any sine substitution For $\int \frac{1}{\sqrt{3}}\,d\theta$ *M1 dependent on first M1*

11.

(b)(i)	$y = \arctan\frac{x}{a}$ $\Rightarrow x = a\tan y$ $\Rightarrow \frac{dx}{dy} = a\sec^2 y$ $\Rightarrow \frac{dx}{dy} = a(1+\tan^2 y)$ $\Rightarrow \frac{dy}{dx} = \frac{a}{a^2+x^2}$	 M1 A1 A1 A1 (ag) **4**	 (a) tan y = and attempt to differentiate both sides Or $\sec^2 y \frac{dy}{dx} = \frac{1}{a}$ Use $\sec^2 y = 1 + \tan^2 y$ o.e. www SC1: Use derivative of arctan x and Chain Rule (properly shown)
(ii)(A)	$\int_{-2}^{2} \frac{1}{4+x^2}dx = \left[\frac{1}{2}\arctan\frac{x}{2}\right]_{-2}^{2}$ $= \frac{\pi}{4}$	M1 A1 A1 **3**	arctan alone, or any tan substitution $\frac{1}{2}$ and $\frac{x}{2}$, or $\int\frac{1}{2}d\theta$ without limits Evaluated in terms of π
(ii)(B)	$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{4}{1+4x^2}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\frac{1}{4}+x^2}dx$ $= \left[2\arctan(2x)\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ $= \pi$	M1 A1 A1 **3**	arctan alone, or any tan substitution 2 and $2x$, or $\int 2d\theta$ without limits Evaluated in terms of π **19**

12.

(a)(i)		G1 G1 G1 **3**	r increasing with θ Correct for $0 \le \theta \le \pi/3$ (ignore extra) Gradient less than 1 at O
(ii)	Area $= \int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta = \frac{1}{2} a^2 \int_0^{\frac{\pi}{4}} \tan^2 \theta \, d\theta$	M1	Integral expression involving $\tan^2\theta$
	$= \frac{1}{2} a^2 \int_0^{\frac{\pi}{4}} \sec^2 \theta - 1 \, d\theta$	M1	Attempt to express $\tan^2\theta$ in terms of $\sec^2\theta$
	$= \frac{1}{2} a^2 [\tan\theta - \theta]_0^{\frac{\pi}{4}}$	A1	$\tan\theta - \theta$ and limits 0, $\frac{\pi}{4}$
	$= \frac{1}{2} a^2 \left(1 - \frac{\pi}{4}\right)$	A1	A0 if e.g. triangle − this answer
		G1 **5**	Mark region on graph